Reconstruction of medical images from CT and SPECT

a mathematician's point of view

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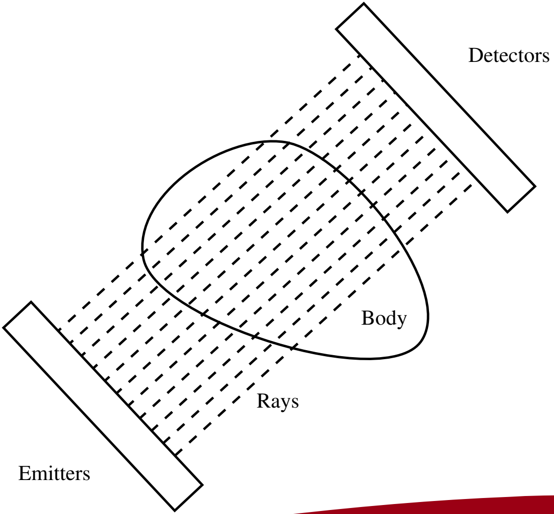
December 17, 2012

Axial Tomography machines provide result of some circular acquisitions. We discuss algorithms of reconstruction from projections and their efficiency in a few cases

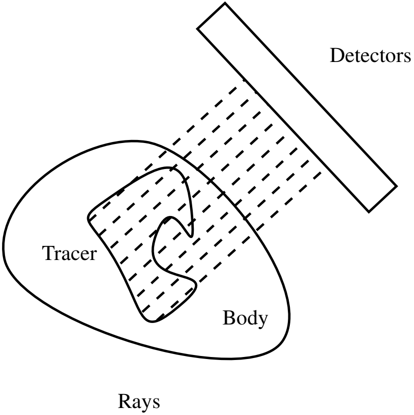
- transmission tomography (e.g. CT): an electromagnetic ray passes trough the patient and is detected at the exit in order to get a morphological analysis of its interior.
- emission tomography (e.g. PET, SPECT): a radioactive tracer is injected into the patient and detected by the machine in order to make an internal functional analysis of the organs.
- hybrid tomography (e.g. SPECT/CT, SPECT/MRI): two simultaneous analysis.

We will focus on CT, SPECT and SPECT/CT problems.

Transmission tomography



Emission tomography

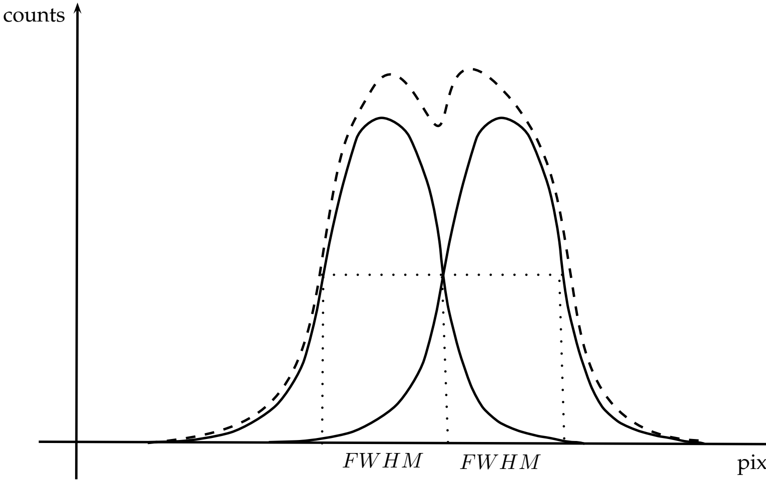


Resolution

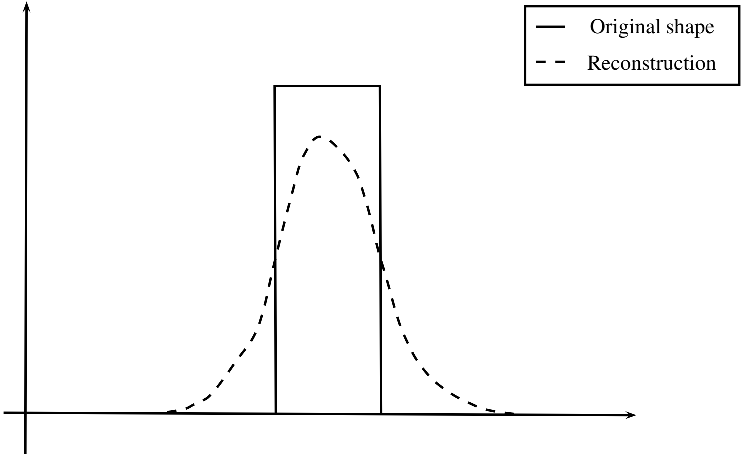
$$R_c \cong D + x\frac{D}{L_{eff}}$$

The resolution of the machine depends on the collimator resolution and on the intrinsic resolution (the resolution of the crystal and the electronics).

$$R_s = \sqrt{R_c^2 + R_i^2}$$


counts
$FWHM$
$FWHM$
pix

Partial volume effect



If a ray passes through a body, it will be subject to attenuation. The Beer's law says that if $I(x)$ is the intensity of a ray and $A(x)$ the attenuation coefficient of the point x, then

$$\frac{\Delta I}{\Delta x} = -A(x)I(x)$$

that is, by integrating:

$$\int_{x_0}^{x_1} A(x)dx = -\int_{x_0}^{x_1} \frac{dI}{I} = -ln(I(x_1) - I(x_0)).$$

In transmission tomography, in 2 dimensions, the Beer's law corresponds to the Radon transform (RT), defined as the integral over a line

$$\mathcal{R}f(t,\theta) = \int_{\ell_{(t,\theta)}} f = \int_{\mathbb{R}^2} f(\bar{x})\delta(t - \bar{x}\cdot\bar{\theta})\, d\bar{x}$$

with $\bar{x} = (x, y)$, $\bar{\theta} = (\cos\theta, \sin\theta)$

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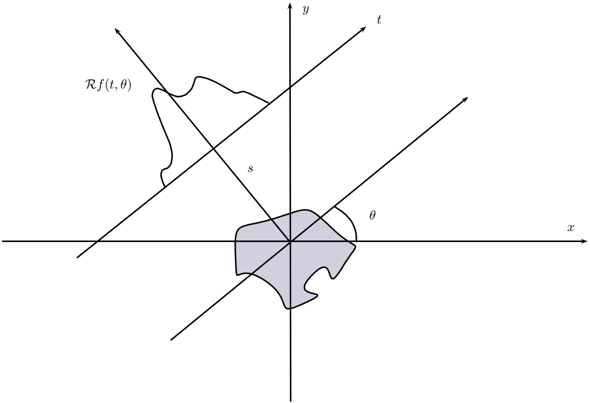
with $\bar{x} = (x, y)$, $\bar{\theta} = (\cos\theta, \sin\theta)$ or equivalently:

$$\mathcal{R}f(t,\theta) = \int_{\mathbb{R}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds = \int_{\mathbb{R}} f(t\bar{\theta} + s\bar{\theta}^{\perp})\, ds$$

with $\bar{\theta}^{\perp} = (-\sin\theta, \cos\theta)$.



$$\mathcal{R}f(t,\theta) = \int_{\mathbb{R}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds$$

Now we ask for every point (x, y) which is the average of the rays that pass through that point. This question is answered by the adjoint operator to $\mathcal{R}$, called Backprojection operator

$$\mathcal{R}^* g(x, y) = \frac{1}{|S^1|} \int_{S^1} g(x \cos\theta + y \sin\theta, \theta)\, d\theta =$$

$$\frac{1}{|S^1|} \int_{\mathbb{R}\times S^1} g(s, \theta)\delta(s - \bar{x} \cdot \bar{\theta})\, ds d\theta$$

where $S^1 = [0, \pi]$ or $S^1 = [0, 2\pi]$

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Warning

$\mathcal{R}^*$ is not the inverse transform of $\mathcal{R}$. In fact $\mathcal{R}^* \mathcal{R} f(\bar{x}) = \frac{2}{|\bar{x}|} * f$

Why π or 2π?

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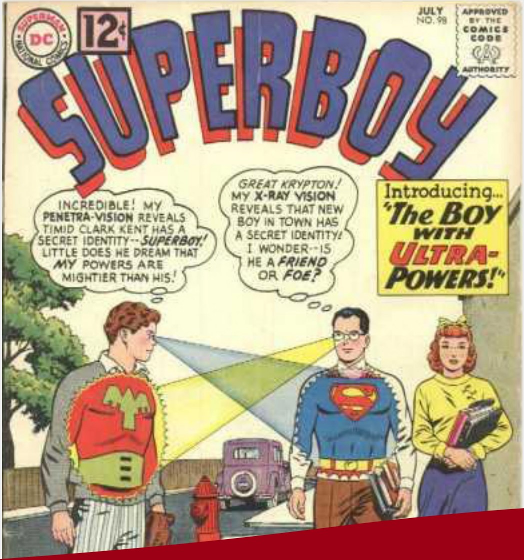
12¢
SUPERBOY
JULY NO. 98
APPROVED BY THE COMICS CODE AUTHORITY
INCREDIBLE! MY PENETRA-VISION REVEALS TIMID CLARK KENT HAS A SECRET IDENTITY--SUPERBOY! LITTLE DOES HE DREAM THAT MY POWERS ARE MIGHTIER THAN HIS!
GREAT KRYPTON! MY X-RAY VISION REVEALS THAT NEW BOY IN TOWN HAS A SECRET IDENTITY! I WONDER--IS HE A FRIEND OR FOE?
Introducing... "The BOY WITH ULTRA-POWERS!"

In the case of emission tomography things are more complicated. From Beer's law we obtain that

$$I(x_1) = I(x_0)\exp\left(-\int_{x_0}^{x_1} A(x)dx\right)$$

Suppose to know the attenuation coefficient, say $a(\bar{x})$, we want to obtain the radioactivity $f(\bar{x})$ by its angular projections.

We define the attenuated Radon transform (briefly AtRT)

$$\mathcal{R}_a f(t,\theta) = \int_{\ell_{(t,\theta)}} e^{-\mathcal{D}a(\bar{x},\theta+\pi)} f(\bar{x})$$

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$$\mathcal{R}_a f(t,\theta) = \int_{\ell_{(t,\theta)}} e^{-\mathcal{D}a(\bar{x},\theta+\pi)} f(\bar{x})$$

where $\mathcal{D}$ is the Divergent beam transform defined as follows

$$\mathcal{D}h(\bar{x},\theta) = \int_0^{+\infty} h(x + t\cos\theta, y + t\sin\theta)\, dt = \int_0^{+\infty} h(\bar{x} + t\bar{\theta})\, dt$$

Then in the CT case we have to solve the following problem

RT problem

$$\text{Given } g \text{ projection data}$$
$$\text{find } f \text{ such that } \mathcal{R}f = g$$

while in SPECT case we can approximate f with the solution of the previous problem

Then in the CT case we have to solve the following problem

RT problem

Given g projection data
find f such that $\mathcal{R}f = g$

while in SPECT case we can approximate f with the solution of the previous problem or solve

AtRT problem

Given g projection data and a attenuation map
find f such that $\mathcal{R}_a f = g$

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Given g projection data and a attenuation map
find f such that $\mathcal{R}_a f = g$

to estimate the attenuation map we may need a simultaneous CT tomography $\longrightarrow$ SPECT/CT.

Theorem (Inversion of the Radon transform)

$$f = \frac{1}{2}\mathcal{R}^* \left[\mathcal{F}^{-1}(|\nu|\mathcal{F}(\mathcal{R}f))\right]$$

where we mean that the direct and inverse Fourier transform is applied only to the variable t.

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where we mean that the direct and inverse Fourier transform is applied only to the variable t.

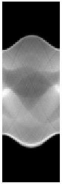
As known the previous formula is numerically inaccurate. Then we use $w(\nu) = p(\nu)|\nu|$ instead of $|\nu|$, with p a low-pass filter, getting the approximated

Filtered Back Projection formula (FBP)

$$f \cong \frac{1}{2}\mathcal{R}^* \left[\mathcal{F}^{-1}(w(\nu)\mathcal{F}(\mathcal{R}f))\right]$$

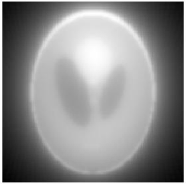
Original phantom f(x,y)

sinogram: Radon transorm Rf(s,theta)



Filtered backprojection

Unfiltered backprojection



Theorem (Error estimate)

Let $f \in C_0^\infty(B(0,1))$ be a b-band-limited function, and let $g = \mathcal{R}f$ be reliably sampled. Let $\tilde{f}$ be the FBP reconstruction, then

$$\|f - \tilde{f}\|_{L^\infty(\mathbb{R}^2)} \leq 2|S^1| \ \|w_b\|_{L^1(\mathbb{R})}\|g - \tilde{g}\|_{L^\infty(\mathbb{R}\times S^1)} + |e_3|$$

with e_3 the quadrature error of the backprojection integral.

Why these assumptions?

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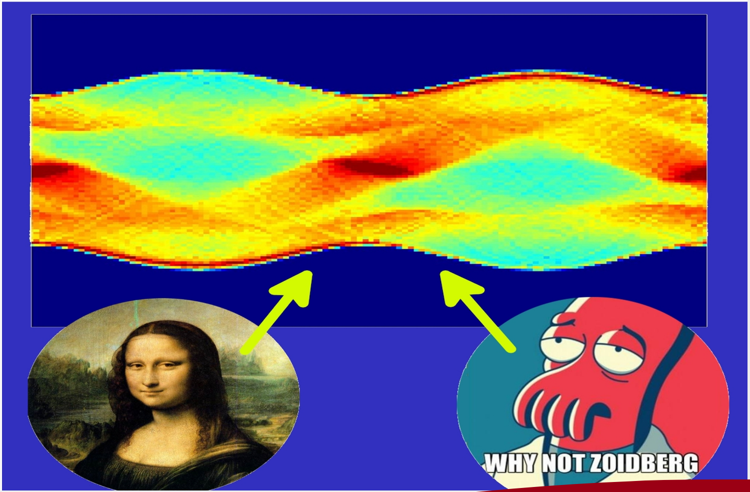


Also the attenuated transform has an inversion formula. Let by the following definitions:

Definition

Let $g(t)$ be a suitable function, then its Hilbert transform is the function

$$\mathcal{H}g(s) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g(t)}{s-t}\, dt$$

where the integral a Cauchy principal value.

Definition

Let us define the function

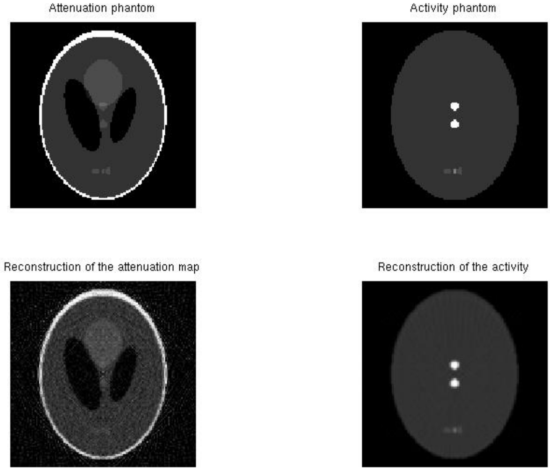
$$h := \frac{1}{2}(I + i\mathcal{H})\mathcal{R}a$$

Theorem (Novikov-Natterer formula)

Let f be a transformable function $g = \mathcal{R}_a f$, and h as in the previous slide. Assume $a(\bar{x})$ known, then f is uniquely determined by the following formula

$$f(\bar{x}) = \frac{1}{4\pi}\mathfrak{Re}\ div \int_{S^1} \theta e^{\mathcal{D}a(\bar{x},\theta+\frac{\pi}{2})}(e^{-h}\mathcal{H}e^{h}g)_{(\bar{x}\cdot\bar{\theta},\theta)}\, d\theta$$

where $S^1 = [0, 2\pi]$.


Attenuation phantom
Activity phantom
Reconstruction of the attenuation map
Reconstruction of the activity

Given a basis of functions $\{b_i(\bar{x})\}_{i=1\ldots n}$ that interpolates the function f, i.e. such that

$$f(\bar{x}) = \sum_{i=0}^{n} c_i b_i(\bar{x}) \qquad \forall \bar{x} \in X$$

where X is properly chosen, then for the linearity of the Radon transform

$$\mathcal{R}f(\bar{y}) = \sum_{i=0}^{n} c_i \mathcal{R}b_i(\bar{y}) \qquad \forall \bar{y} \in Y = \{(t_j, \theta_j)\}.$$

This is equivalent to the solution of

$$Af = c$$

where $A(i,j) = \mathcal{R}b_i(t_j,\theta_j)$ is a matrix $N^2 \times lp$, f is the unknown vector such that $f_i = f(x_i)$ and c is the vector of the projection data $c_j = \mathcal{R}f(t_j,\theta_j)$.
The methods using this approach are known as Algebraic Reconstruction Techniques or ART.

Using the ART approach we may have to face some problems:

- underdetermined $\longrightarrow$ least squares
- ill-conditioned $\longrightarrow$ regularization
- huge $\longrightarrow$ sparseness

We choose the *natural pixel basis*

$$b_i(x, y) = \chi_{P_i}(x, y).$$

with P_i the i-th pixel of the reconstructed image. We know the Radon transform of each one of these items

$$\mathcal{R}b_i(t, \theta) = meas(\ell_{t,\theta} \cap P_i)$$

where $\ell_{t,\theta} = \{t\bar{\theta} + s\bar{\theta}^{\perp} \mid s \in \mathbb{R}\}$.

Since the matrix A is large and sparse, we can solve the system $Af = c$ by iterative methods.

- the initial vector $f^{(0)}$ is a *blank*.
- the image at step k, $f^{(k)}$, is projected and compared with the data.
- the image is modified considering the error found in the previous step.

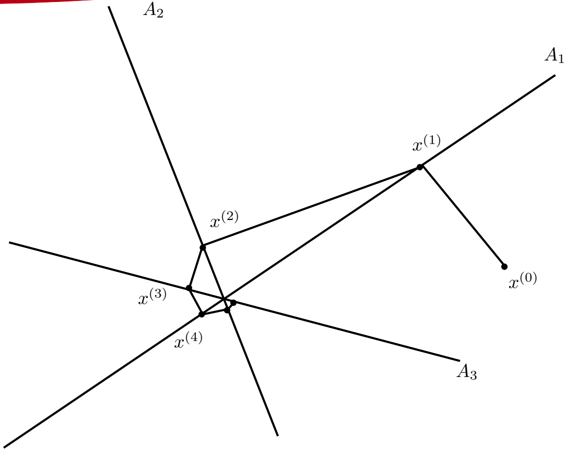
The following methods are the most popular

The Kaczmarz method projects the vector $f^{(k)}$ on k-th row of A.

$$f^{(k+1)} = f^{(k)} + \frac{c_i - A_i^T \cdot f^{(k)}}{A_i^T \cdot A_i} A_i$$

to increase the speed we use

$$f^{(k+1)} = f^{(k)} + \lambda_k \frac{c_i - A_i^T \cdot f^{(k)}}{A_i^T \cdot A_i} A_i$$



$$f^{(k+1)} = f^{(k)} + \frac{c_i - A_i^T \cdot f^{(k)}}{A_i^T \cdot A_i} A_i$$

Maximum Likelihood Expectation Maximization (MLEM) is based on a probabilistic argument (the noise is assumed to be Poissonian).

$$L(f) = P(c|f) = \prod_{i=1}^{lp} \frac{(c_i^*)^{c_i}}{c_i!} e^{-c_i^*}$$

where $c^* = Af$ is the exact sinogram, i.e. the projection of the exact solution f. Equivalently, it maximizes

$$l(f) = \log(L(f)) = \sum_{i=1}^{lp} -(Af)_i + C_i \log(Af)_i + K$$

scheme in a vectorial and multi-step form

1. $c^f := Af^{(k)}$
2. $c^q := c./c^f$ (punctual division)
3. $f^b = A^T c^q$
4. $s_j = \sum_i a_{i,j}$
5. $f^{(k+1)} = f^{(k)}.*f^b./s$ (product and division are made elementwise)

Least Squares Conjugate Gradient (LSCG) is the conjugate gradient method applied to the normal equation of the problem.
Initialization phase:

- given $f^{(0)}$ initial value
- $s^{(0)} = c - Af^{(0)}$
- $r^{(0)} = p^{(0)} = A^T s^{(0)}$
- $q^{(0)} = Ap^{(0)}$

- $\alpha = \frac{|r^{(k)}|^2}{|q^{(k)}|^2}$
- $f^{(k+1)} = f^{(k)} + \alpha p^{(k)}$
- $s^{(k+1)} = s^{(k)} - \alpha q^{(k)}$
- $r^{(k+1)} = A^T s^{(k)}$
- $\beta = \frac{|r^{(k+1)}|^2}{|r^{(k)}|^2}$
- $p^{(k+1)} = r^{(k+1)} + \beta p^{(k)}$
- $q^{(k+1)} = Ap^{(k+1)}$

LSCG example

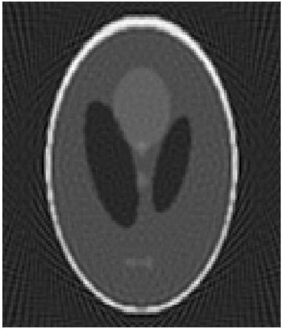
Now instead of the natural pixel basis we can use another basis:

$$b_i(\bar{x}) = B(\bar{x} - \bar{x}_i) = K(\bar{x}, \bar{x}_i)$$

with B a (essentially) compact supported and radial.

Function name	f
Ball	$\chi_{B(\frac{1}{\varepsilon},0)}(r)$
Gaussian	$e^{-\varepsilon^2 r^2}$
Wendland $\varphi_{2,0}$	$(1-\varepsilon r)_+^2$
Wu $\psi_{1,1}$	$(1-\varepsilon r)_+^2(\varepsilon r + 2)$

with $r = \|x\|_2$.



Original phantom and kernel reconstruction with Gaussian kernel and shape parameter $\varepsilon = 1$, after 50 iterations of LSCG.

Lemma

If $\phi(x) = \varphi(\|x\|)$ is a radial function, then its Radon transform $\mathcal{R}f$ is readial, i.e. it depends only on t and it is even.

Theorem

If $\phi(x-y) = K(x,y)$ is a radial function, $\phi \in L^1(\mathbb{R}^d)$, continuous, bounded and positive definite on $\mathbb{R}^2$, then its Radon transform $\mathcal{R}f(t)$ is bounded and positive definite on $\mathbb{R}^1$, provided $\mathcal{R}f \in L^1(\mathbb{R})$.

Theorem (Interpolation error bound for Kernel method)

Let $f \in C_0^\infty(B(0,1))$ a b-band-limited function, and let $g = \mathcal{R}f$ be reliably sampled. Let K be the interpolating Kernel function such that $\mathcal{R}K$ is a symmetric and strictly positive definite kernel and its domain Ω be such that $\partial\Omega$ has regularity at least C^1.
Then there exist positive constants h_0 and $\tilde{C}$ such that, if $h_{X,\Omega} \leq h_0$, then

$$\|f(\cdot) - \sum_{i=0}^{n} c_i K_i(\cdot)\|_{L^\infty} \leq 2|S^1|\; b\sqrt{\frac{N}{18}}\; \tilde{C} h_{X,\Omega} \|\mathcal{R}f\|_{\mathcal{N}_{\mathcal{R}K}(\Omega)}$$

with $h_{X,\Omega}$ the meshsize.

Let us consider, as before, the basis $b_i(\bar{x}) = \chi_{P_i}$ and assume

$$f(\bar{x}) = \sum_{i=0}^{n} c_i b_i(\bar{x})$$

then for linearity

$$\mathcal{R}_a f(\bar{y}) = \sum_{i=0}^{n} c_i \mathcal{R}_a b_i(\bar{y}) \qquad \forall \bar{y} \in Y = \{(t_j, \theta_j)\}$$

i.e.

$$Bc = d$$

where $B_{i,j} = \mathcal{R}_a b_i(t_j, \theta_j)$, $c_i = f(\bar{x}_i)$ the unknown term and $d_j = \mathcal{R}_a f(t_j, \theta_j)$ the data.

In order to compute the matrix B we have to consider the attenuation coefficients in the natural pixel basis

$$a(\bar{x}) = \sum_{k=1}^{N^2} g_k \chi_{P_k}(\bar{x}).$$

According to Beer's law

$$I_{out} = I_{in} \ \exp\left(- \sum_{k=1}^{N^2} g_k \ \ meas(P_k \cap \ell^+_{\bar{x},\theta}) \right).$$

Now, if we consider the matrix A used in CT tomography we can compute the outgoing rays from the pixel P_i in (t_j, θ_j) as

$$B_{i,j} = A_{i,j}\ \exp\left(-\sum_{(k_1,k_2)\in K_{(i,j)}} g_k\ \ meas(P_k \cap \ell_{t_j,\theta_j})\right) =$$

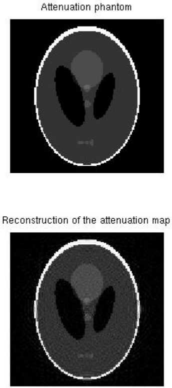
$$A_{i,j}\ \exp\left(-\sum_{(k_1,k_2)\in K_{(i,j)}} g_k\ A_{k_1,k_2}\right)$$

where $K_{(i,j)} = \{(k_1, k_2)\} \subset \{1, \ldots, N^2\}^2$ is the set s.t. $k = lp - ((k_1 - 1)p + k_2) + 1$ are the indexes of the pixels covered by the line $\ell^+_{\bar{x}_i,\theta_j}$.

We can introduce a relaxation parameter $\lambda \in [0, 1]$ to weight the effect of the attenuation

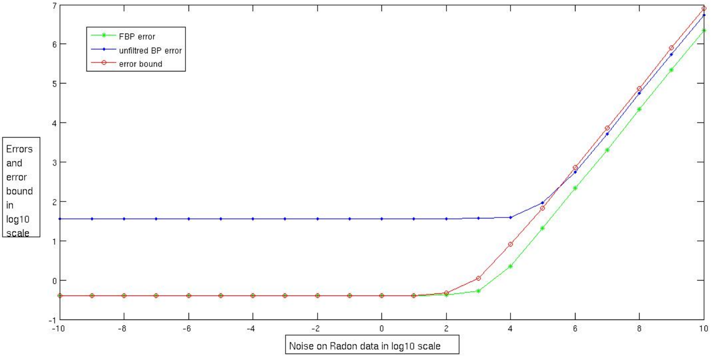
$$B_{i,j}^{(\lambda)} = A_{i,j} \ \exp\left(-\lambda \sum_{(k_1,k_2)\in K_{(i,j)}} g_k \ A_{k_1,k_2}\right)$$

Note that $B^{(0)} = A$ and $B^{(1)} = B$.
We observe that with this little change the linear system is more accurate.

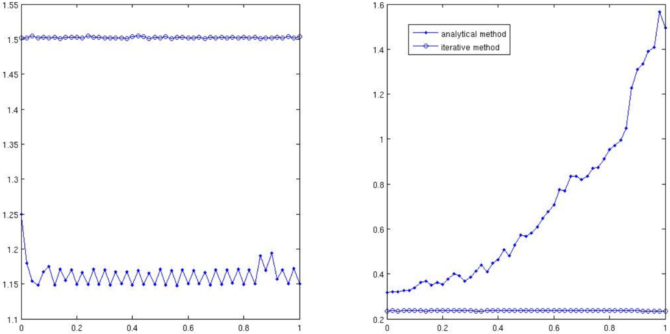


Analytical reconstruction of a SPECT/CT phantom data with $\lambda = 0.1$.

Analytical methods

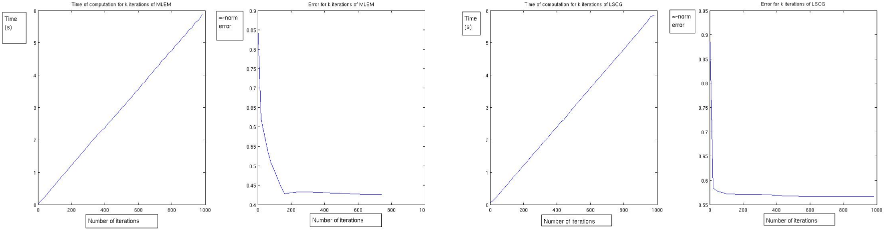


A test for the error bound using the standard “Ram-Lack” filter.

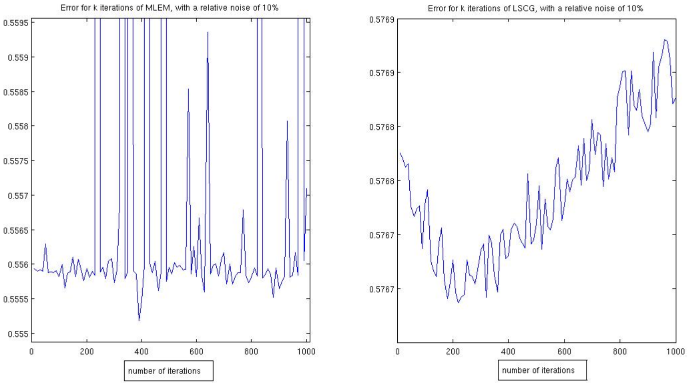


Time of resolution in seconds (left) and error (right) for the analytical and iterative methods for the resolution of the hybrid SPECT/CT simulated problem at several relative noise levels form 0 to 100%.

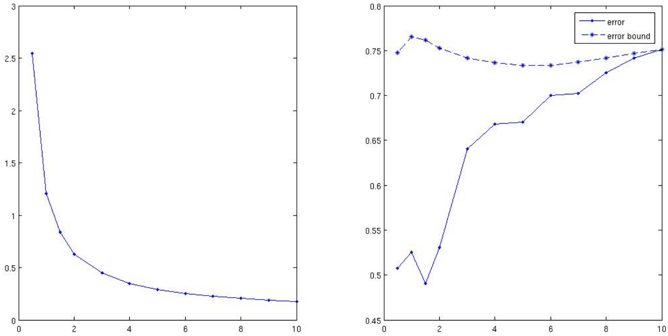
Iterative methods



Time and error test for MLEM (upper) and LSCG (lower) at several iterations.

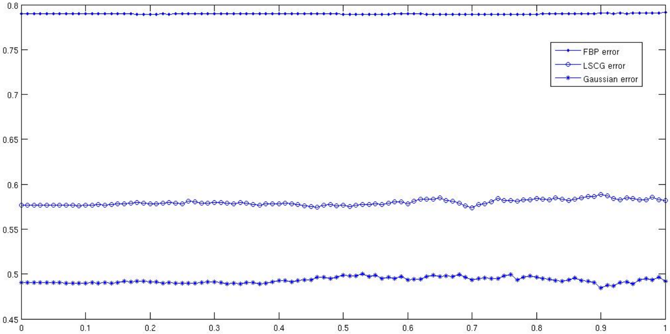


Error of the MLEM and LSCG algorithms with a noise of $\sigma = 10\%$ after several numbers of iterations.

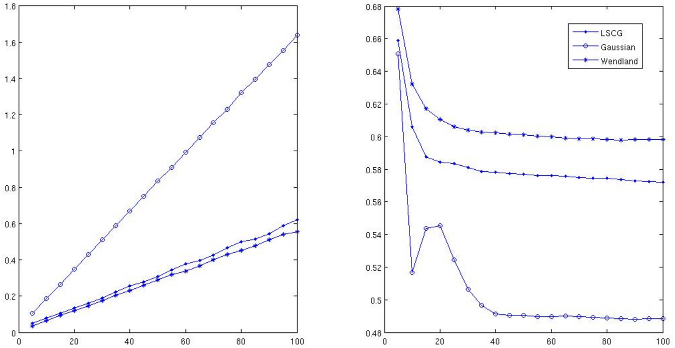


Time of computation in seconds (left), error and error bound (right) in ∞-norm of the kernel method for the functions Gaussian (upper) and “ball” (lower) with several shape parameters $\varepsilon \in [0.5, 10]$.

Comparison



Error of FBP, LSCG, and Gaussian algorithms as the relative error varies from 0 to 100%.



Computational times in seconds (left) and errors (right) for the LSCG and its kernel versions. Along x we have the number of iterations.

Essential bibliography

- Feeman, *The mathematics of medical imaging: A beginners guide*, Springer, 2010
- Hansen P.C., Hansen M., *AIR Tools - A MATLAB Package of Algebraic Iterative Reconstruction Methods*, Journal of Computational and Applied Mathematics, 2011
- Natterer F., *The mathematics of computerized tomography*, SIAM: Society for Industrial and Applied Mathematic, 2001
- Sironi A., *Medical Image Reconstruction Using Kernel Based Methods*, Master's Thesis, University of Padova, 2011
- Toft P., *The Radon transform; theory and implementation*, Ph.D. thesis. Department of Mathematical Modeling, Technical University of Denmark, 1996