Introduction to Analytic Thomography

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May 14, 2013

Calendar of seminars

May 14 Introduction to Analytic Tomography

general introduction, Radon tranform, FBP

May 21 Reconstruction in X-ray computed tomography

iterative methods, kernel methods

May 28 The mathematics behind SPECT/TC reconstruction

analytical and iterative hybrid reconstruction

June 4 Resolution of a Gamma Camera: experimental data and analytical formula

resolution formulas

June 11 An alternative Radon transform for the correction of partial volume effect

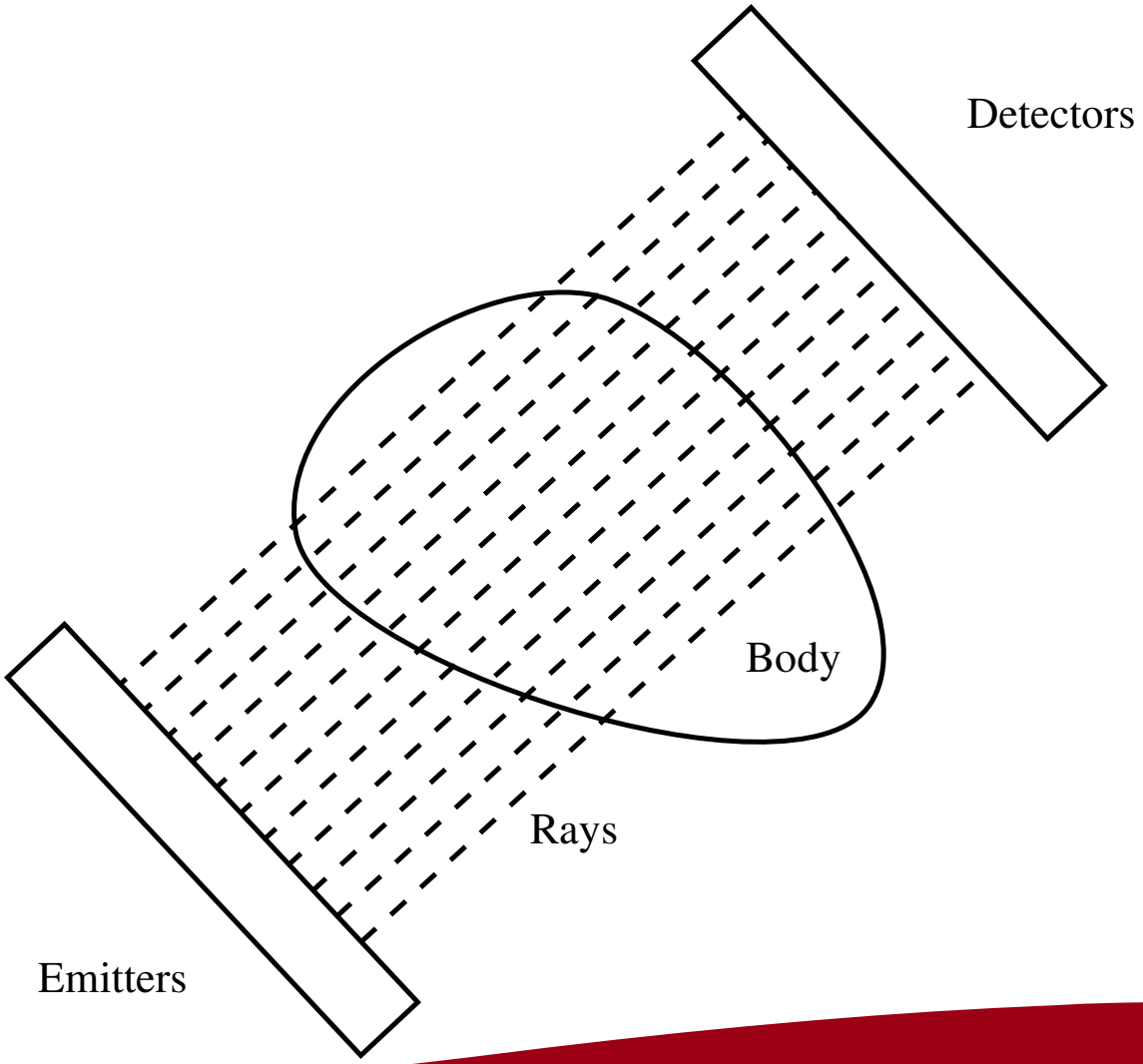
a modest proposal..

Axial Tomography machines provide result of some circular acquisitions. We discuss algorithms of reconstruction from projections and their efficiency in a few cases

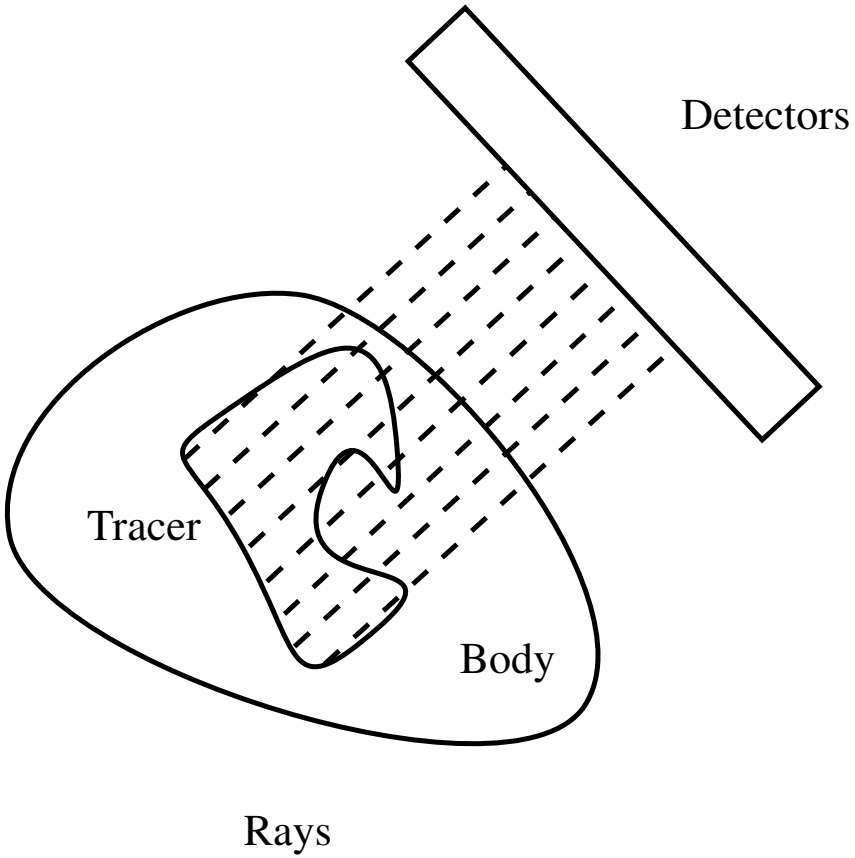
- transmission tomography (e.g. CT): an electromagnetic ray passes trough the patient and is detected at the exit in order to get a morphological analysis of its interior.
- emission tomography (e.g. PET, SPECT): a radioactive tracer is injected into the patient and detected by the machine in order to make an internal functional analysis of the organs.
- hybrid tomography (e.g. SPECT/CT, SPECT/MRI): two simultaneous analysis.

We will focus on CT, SPECT and SPECT/CT problems.

Transmission tomography



Emission tomography

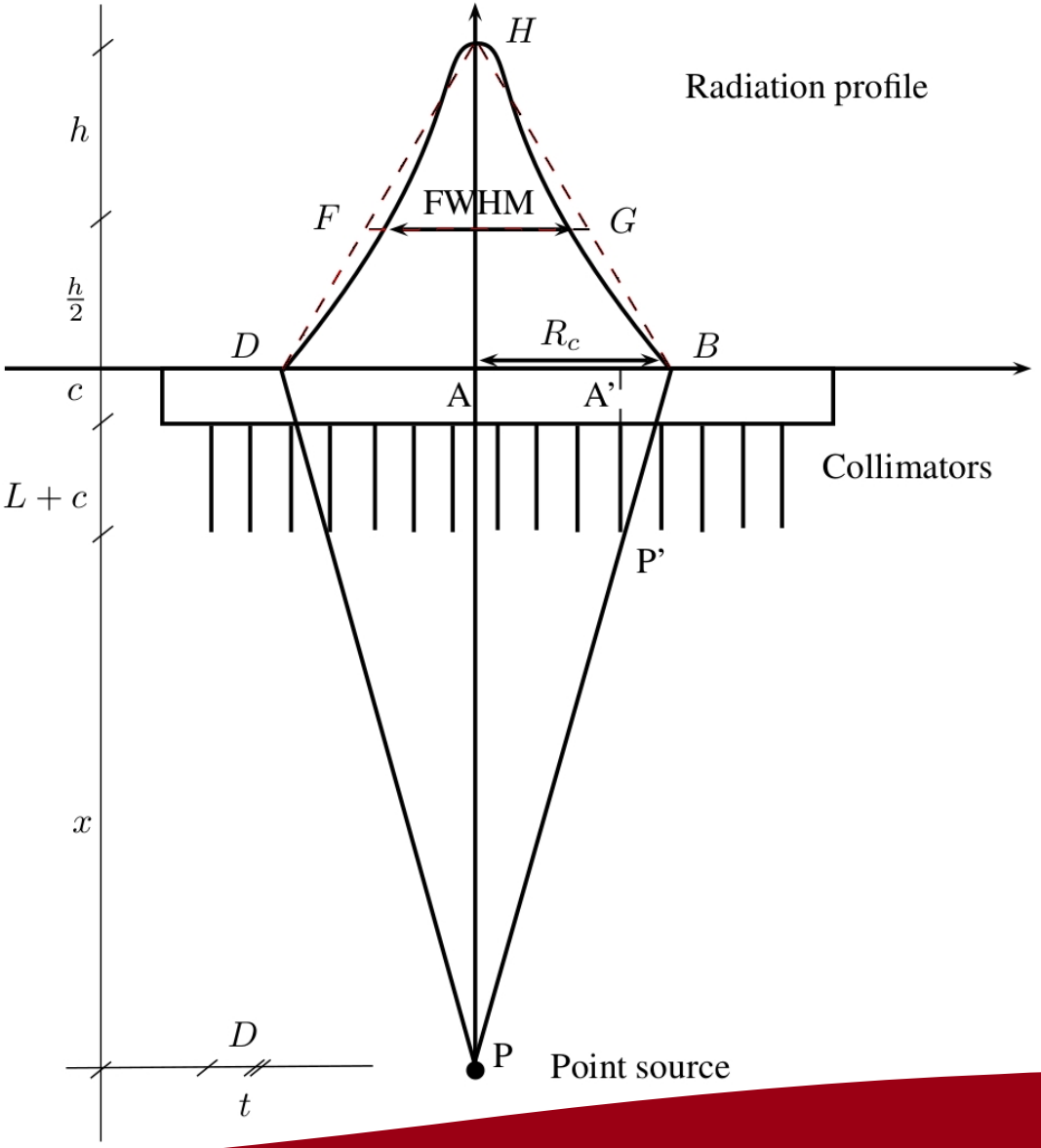


Resolution

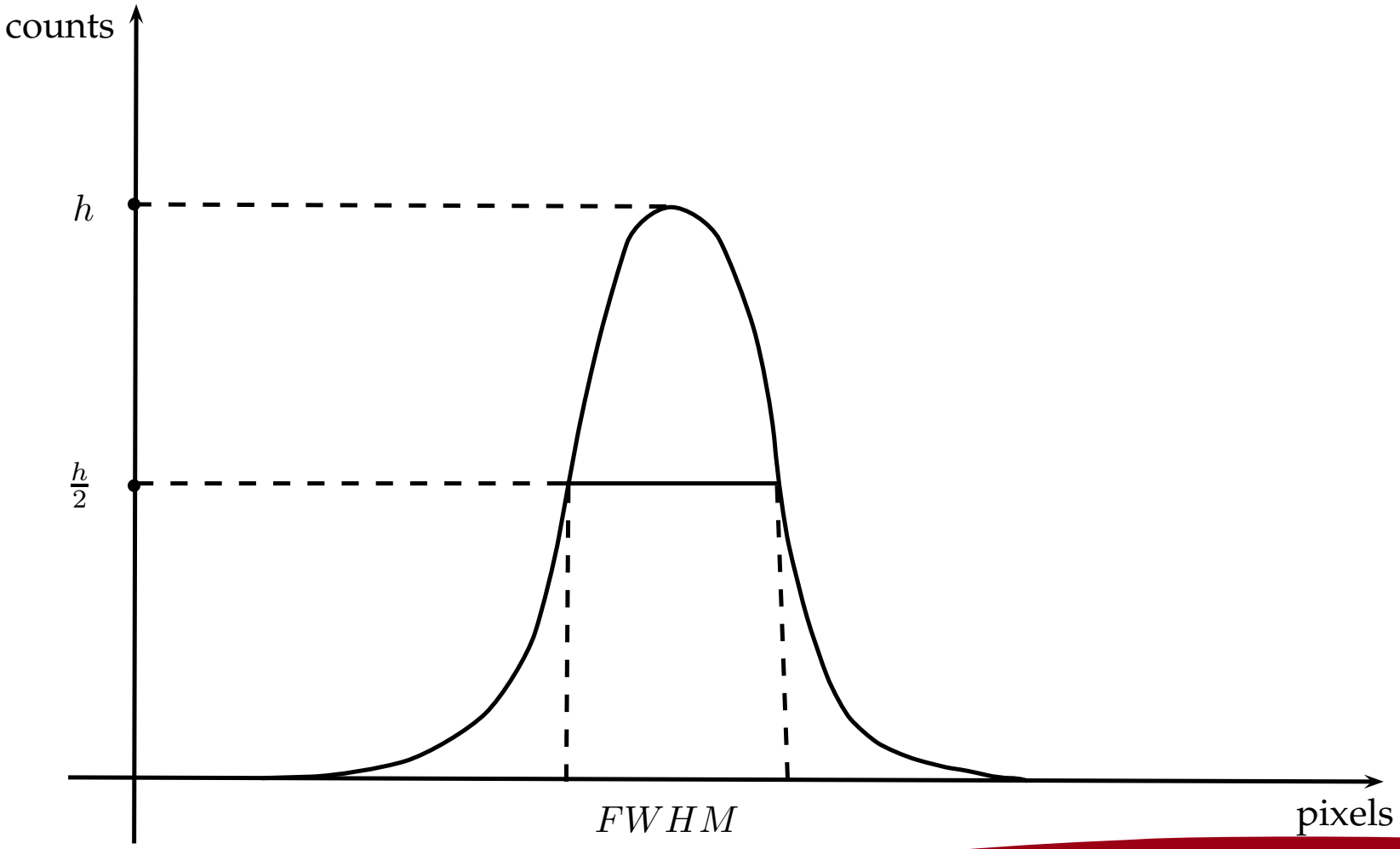
$$R_c = D\left(1 + \frac{x + c}{L_{eff}}\right)$$

The resolution of the machine depends on the collimator resolution and on the intrinsic resolution (the resolution of the crystal and the electronics).

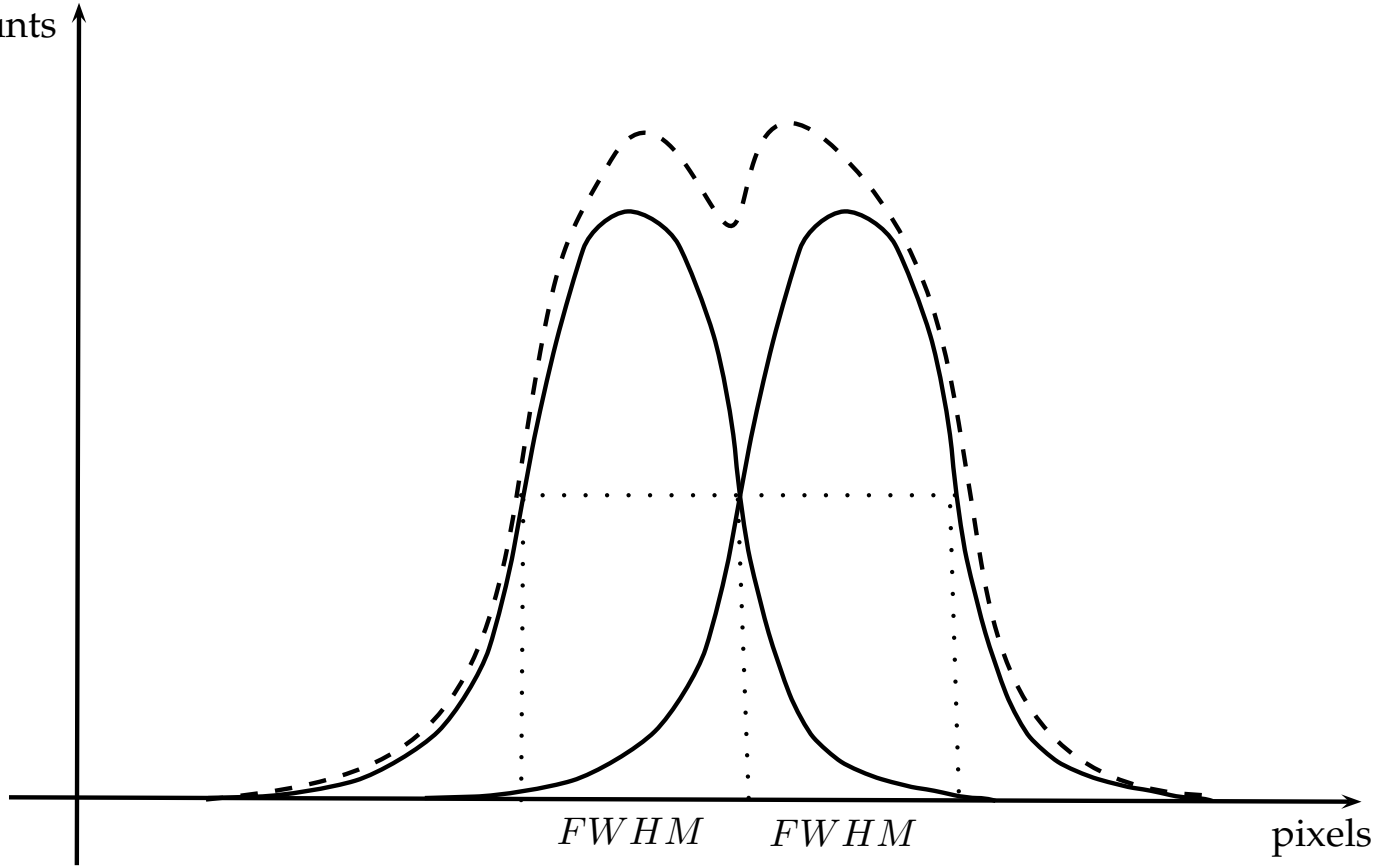
$$R_s = \sqrt{R_c^2 + R_i^2}$$


H
Radiation profile
h
F
FWHM
G
$\frac{h}{2}$
D
R_c
B
c
A
A'
Collimators
$L + c$
P'
x
D
t
P
Point source

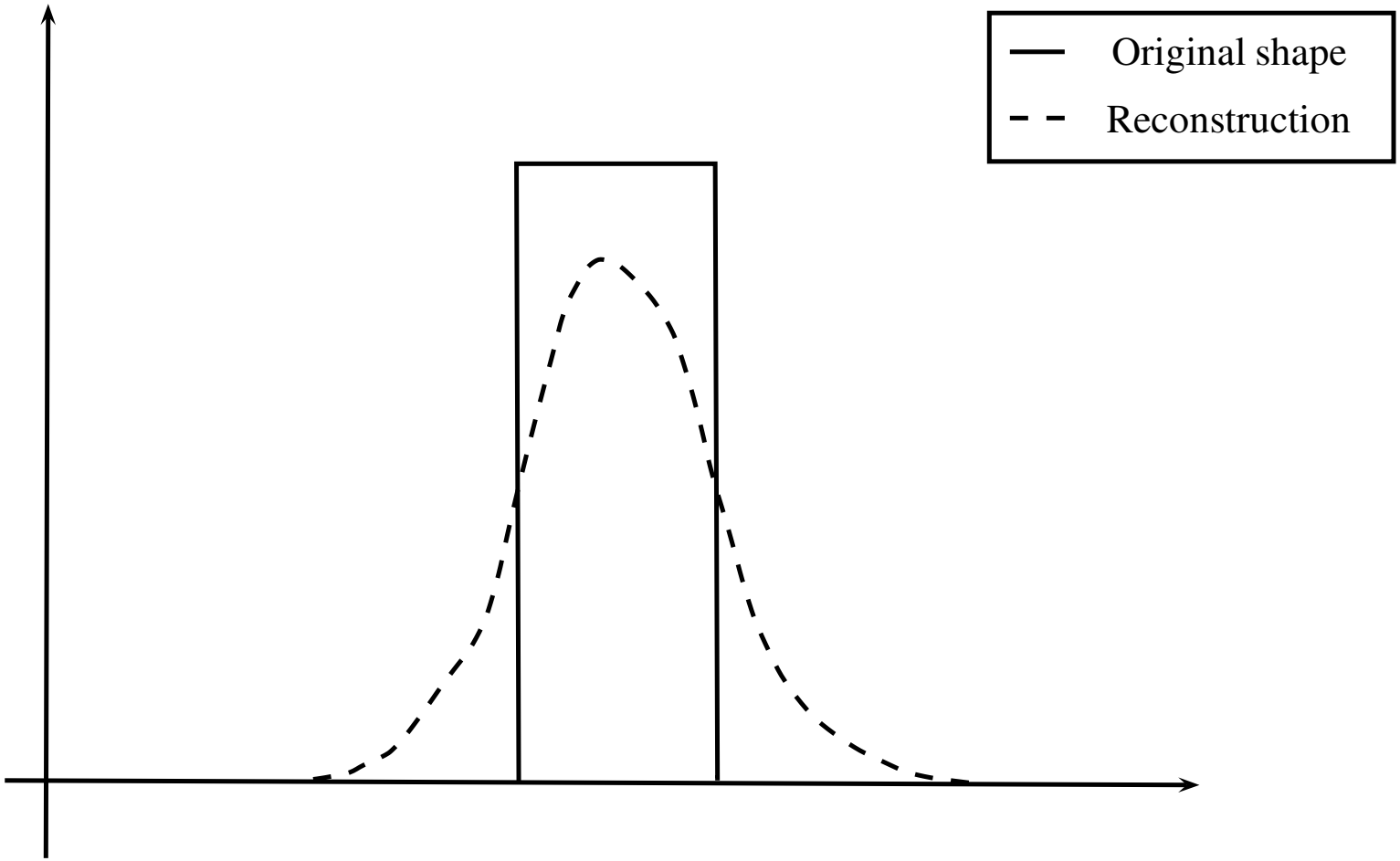
FWHM



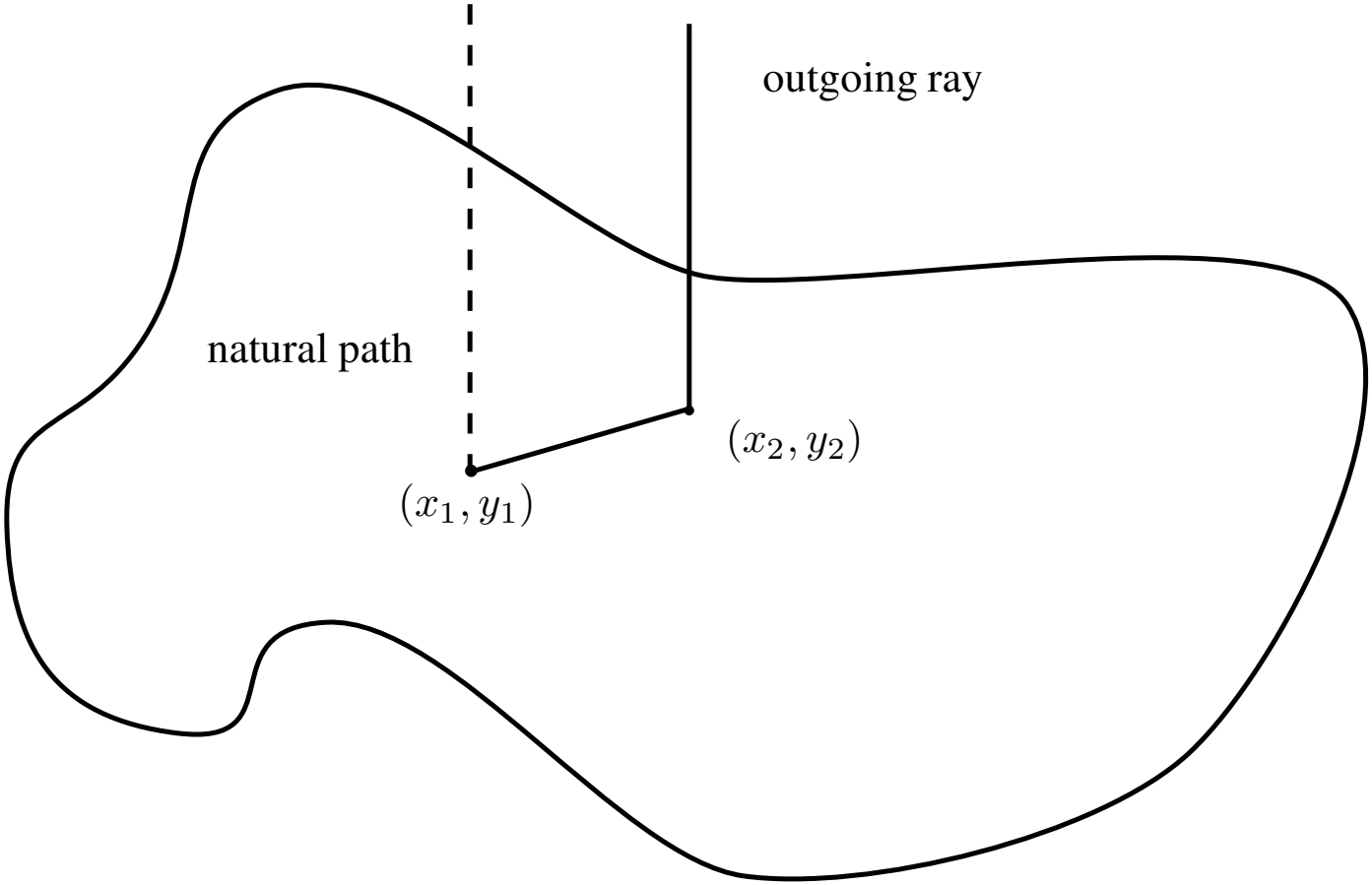
FWHM

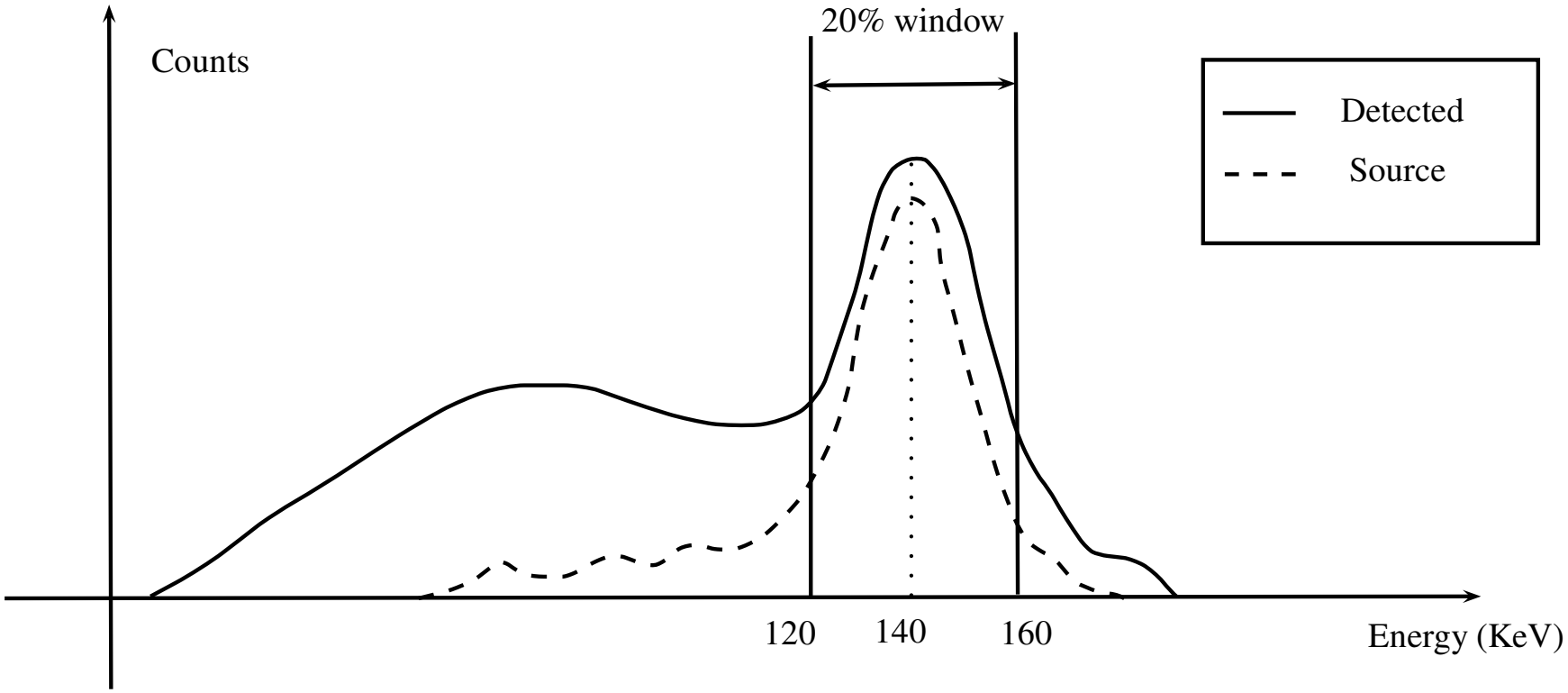


Partial volume effect



Scattering




Counts
20% window
Detected
Source
120
140
160
Energy (KeV)

In order to model correctly the tomography machine we will make some assumptions:

- all rays have no width and we can consider them as straight lines.
- all rays at angle θ are parallel.
- the machine runs I scans at the angles $(\theta_1, \ldots, \theta_I)$; the detectors move on a circle (typically $I = 120, 60$).
- for each scan $j \in 1, \ldots, I$ the machine gets p linear samples (typically $p = N$, or $p = 128$ or $p = 64$).
- the total time of acquisition is relatively short so is indifferent to run an acquisition before or after another one.
- the reconstructed image has dimension $N \times N$ (typically $N = 128$ or $N = 64$).

If a ray passes through a body, it will be subject to attenuation. The Beer's law says that if $I(x)$ is the intensity of a ray and $A(x)$ the attenuation coefficient of the point x, then

$$\frac{\Delta I}{\Delta x} = -A(x)I(x)$$

that is, by integrating:

$$\int_{x_0}^{x_1} A(x)dx = -\int_{x_0}^{x_1} \frac{dI}{I} = -ln(I(x_1) - I(x_0)).$$

In transmission tomography, in 2 dimensions, the Beer's law corresponds to the Radon transform (RT), defined as the integral over a line

$$\mathcal{R}f(t,\theta) = \int_{\ell_{(t,\theta)}} f = \int_{\mathbb{R}^2} f(\bar{x})\delta(t - \bar{x}\cdot\bar{\theta})\, d\bar{x}$$

with $\bar{x} = (x, y)$, $\bar{\theta} = (\cos\theta, \sin\theta)$

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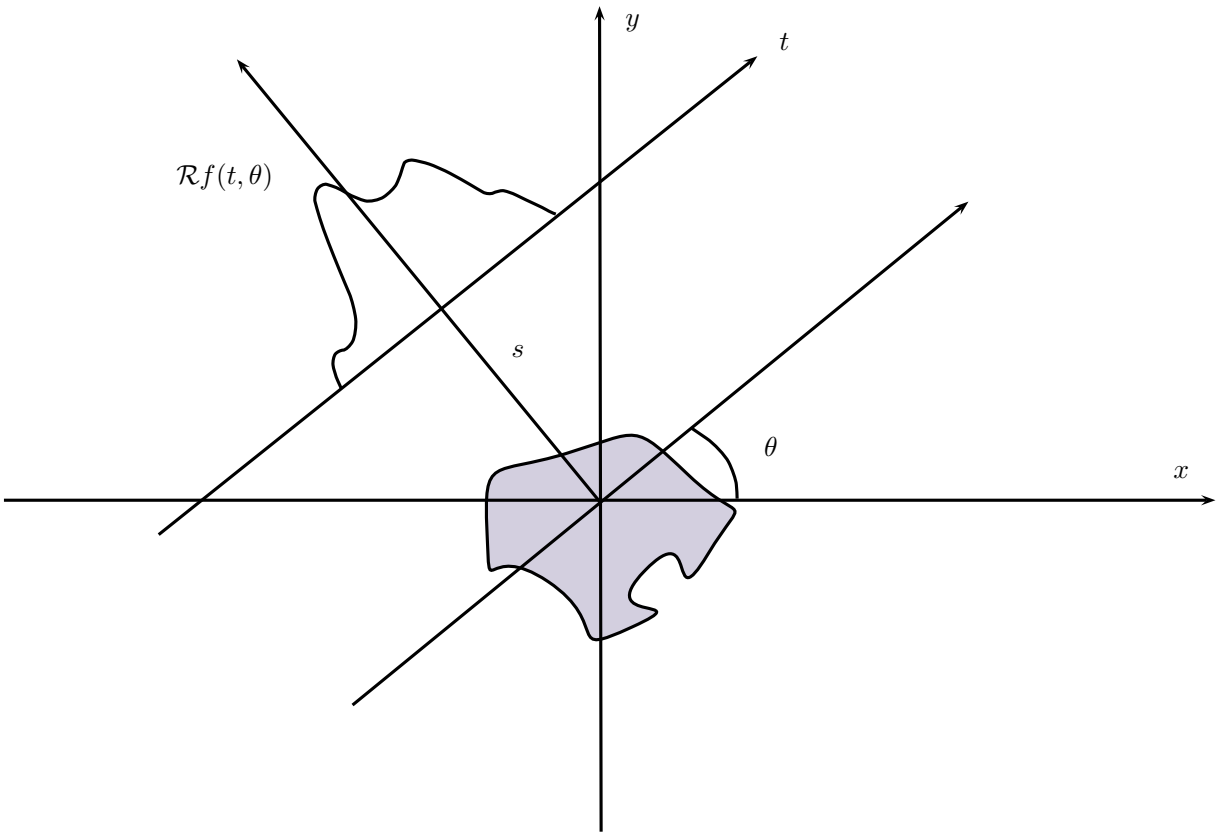
with $\bar{x} = (x, y)$, $\bar{\theta} = (\cos\theta, \sin\theta)$ or equivalently:

$$\mathcal{R}f(t,\theta) = \int_{\mathbb{R}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds = \int_{\mathbb{R}} f(t\bar{\theta} + s\bar{\theta}^{\perp})\, ds$$

with $\bar{\theta}^{\perp} = (-\sin\theta, \cos\theta)$.



$$\mathcal{R}f(t,\theta) = \int_{\mathbb{R}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds$$

Now we ask for every point (x, y) which is the average of the rays that pass through that point. This question is answered by the adjoint operator to $\mathcal{R}$, called Backprojection operator

$$\mathcal{R}^* g(x, y) = \frac{1}{|S^1|} \int_{S^1} g(x \cos\theta + y \sin\theta, \theta)\ d\theta =$$

$$\frac{1}{|S^1|} \int_{\mathbb{R} \times S^1} g(s, \theta) \delta(s - \bar{x} \cdot \bar{\theta})\ ds d\theta$$

where $S^1 = [0, \pi]$ or $S^1 = [0, 2\pi]$

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Warning

$\mathcal{R}^*$ is not the inverse transform of $\mathcal{R}$. In fact $\mathcal{R}^* \mathcal{R} f(\bar{x}) = \frac{2}{|\bar{x}|} * f$

Why π or 2π?

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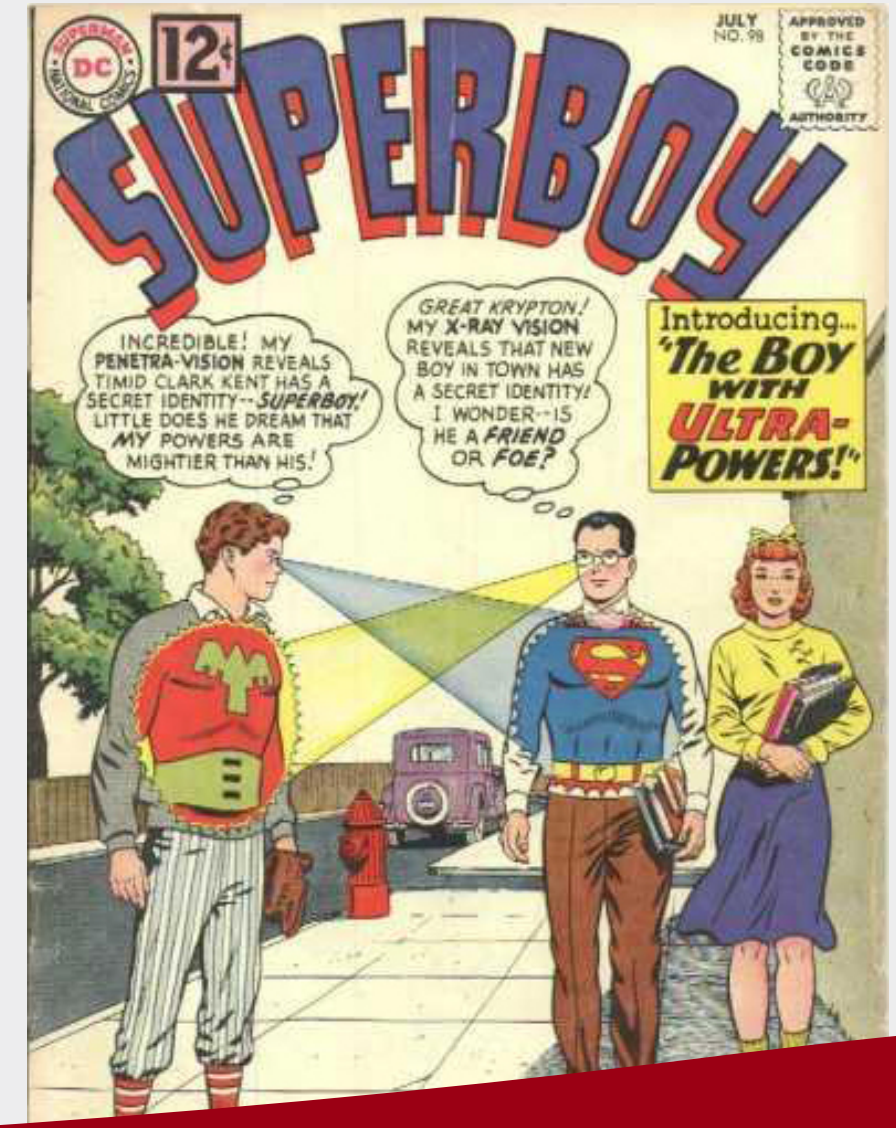
12¢
SUPERBOY
JULY
NO. 98
APPROVED BY THE COMICS CODE AUTHORITY
INCREDIBLE! MY PENETRA-VISION REVEALS TIMID CLARK KENT HAS A SECRET IDENTITY--SUPERBOY! LITTLE DOES HE DREAM THAT MY POWERS ARE MIGHTIER THAN HIS!
GREAT KRYPTON! MY X-RAY VISION REVEALS THAT NEW BOY IN TOWN HAS A SECRET IDENTITY! I WONDER--IS HE A FRIEND OR FOE?
Introducing... "The BOY WITH ULTRA-POWERS!"

In the case of emission tomography things are more complicated. From Beer's law we obtain that

$$I(x_1) = I(x_0) \exp\left(-\int_{x_0}^{x_1} A(x)dx\right)$$

Suppose to know the attenuation coefficient, say $a(\bar{x})$, we want to obtain the radioactivity $f(\bar{x})$ by its angular projections.

We define the attenuated Radon transform (briefly AtRT)

$$\mathcal{R}_a f(t,\theta) = \int_{\ell_{(t,\theta)}} e^{-\mathcal{D}a(\bar{x},\theta+\pi)} f(\bar{x})$$

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where $\mathcal{D}$ is the Divergent beam transform defined as follows

$$\mathcal{D}h(\bar{x},\theta) = \int_0^{+\infty} h(x + t\cos\theta, y + t\sin\theta)\, dt = \int_0^{+\infty} h(\bar{x} + t\bar{\theta})\, dt$$

Then in the CT case we have to solve the following problem

RT problem

$$\text{Given } g \text{ projection data}$$
$$\text{find } f \text{ such that } \mathcal{R}f = g$$

while in SPECT case we can approximate f with the solution of the previous problem

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AtRT problem

$$\text{Given } g \text{ projection data and } a \text{ attenuation map}$$
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AtRT problem

$$\text{Given } g \text{ projection data and } a \text{ attenuation map}$$
$$\text{find } f \text{ such that } \mathcal{R}_a f = g$$

to estimate the attenuation map we may need a simultaneous CT tomography $\longrightarrow$ SPECT/CT.

Central Slice Theorem

Let f be a Fourier and Radon transformable function. Then

$$\mathcal{F}_2 f(T\cos\theta, T\sin\theta) = \mathcal{F}(\mathcal{R}f)(T,\theta)$$

Where we mean for $\mathcal{F}_2$ the 2-dimensional Fourier transform and $\mathcal{F}$ the 1-dimensional Fourier transform applied only to the variable t.

$$\begin{tikzcd}
f \arrow[rr, "\mathcal{R}"] \arrow[dr, "\mathcal{F}_2"'] & & \mathcal{R}f \arrow[dl, "\mathcal{F}_1"] \\
& \hat{f} &
\end{tikzcd}$$

Theorem (Inversion of the Radon transform)

$$f = \frac{1}{2}\mathcal{R}^* \left[\mathcal{F}^{-1}(|\nu|\mathcal{F}(\mathcal{R}f))\right]$$

where we mean that the direct and inverse Fourier transform is applied only to the variable t.

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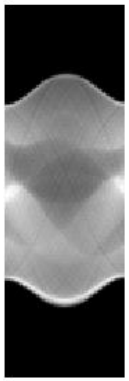
As known the previous formula is numerically inaccurate. Then we use $w(\nu) = p(\nu)|\nu|$ instead of $|\nu|$, with p a low-pass filter, getting the approximated

Filtered Back Projection formula (FBP)

$$f \cong \frac{1}{2}\mathcal{R}^* \left[\mathcal{F}^{-1}(w(\nu)\mathcal{F}(\mathcal{R}f))\right]$$

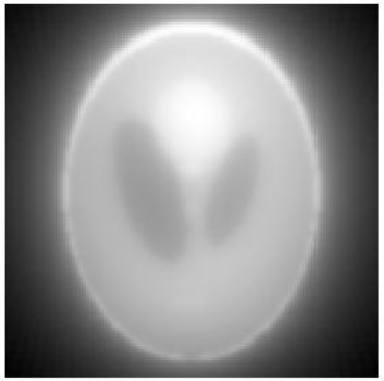
Original phantom f(x,y)

sinogram: Radon transorm Rf(s,theta)



Filtered backprojection

Unfiltered backprojection



Theorem (Error estimate)

Let $f \in C_0^\infty(B(0,1))$ be a b-band-limited function, and let $g = \mathcal{R}f$ be reliably sampled. Let $\tilde{f}$ be the FBP reconstruction, then

$$\|f - \tilde{f}\|_{L^\infty(\mathbb{R}^2)} \leq 2|S^1| \ \ \|w_b\|_{L^1(\mathbb{R})} \|g - \tilde{g}\|_{L^\infty(\mathbb{R}\times S^1)} + |e_3|$$

with e_3 the quadrature error of the backprojection integral.

Why these assumptions?

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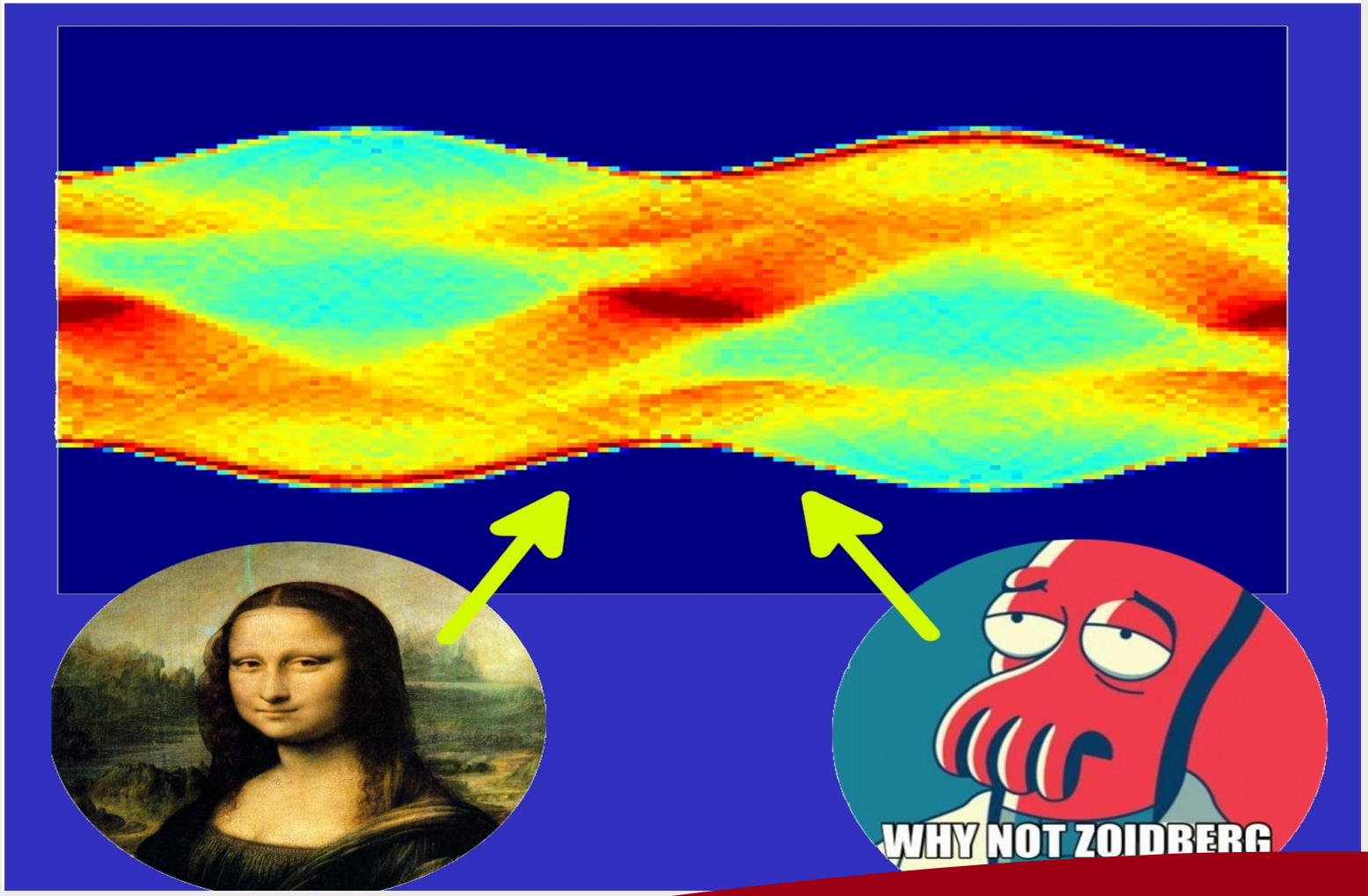
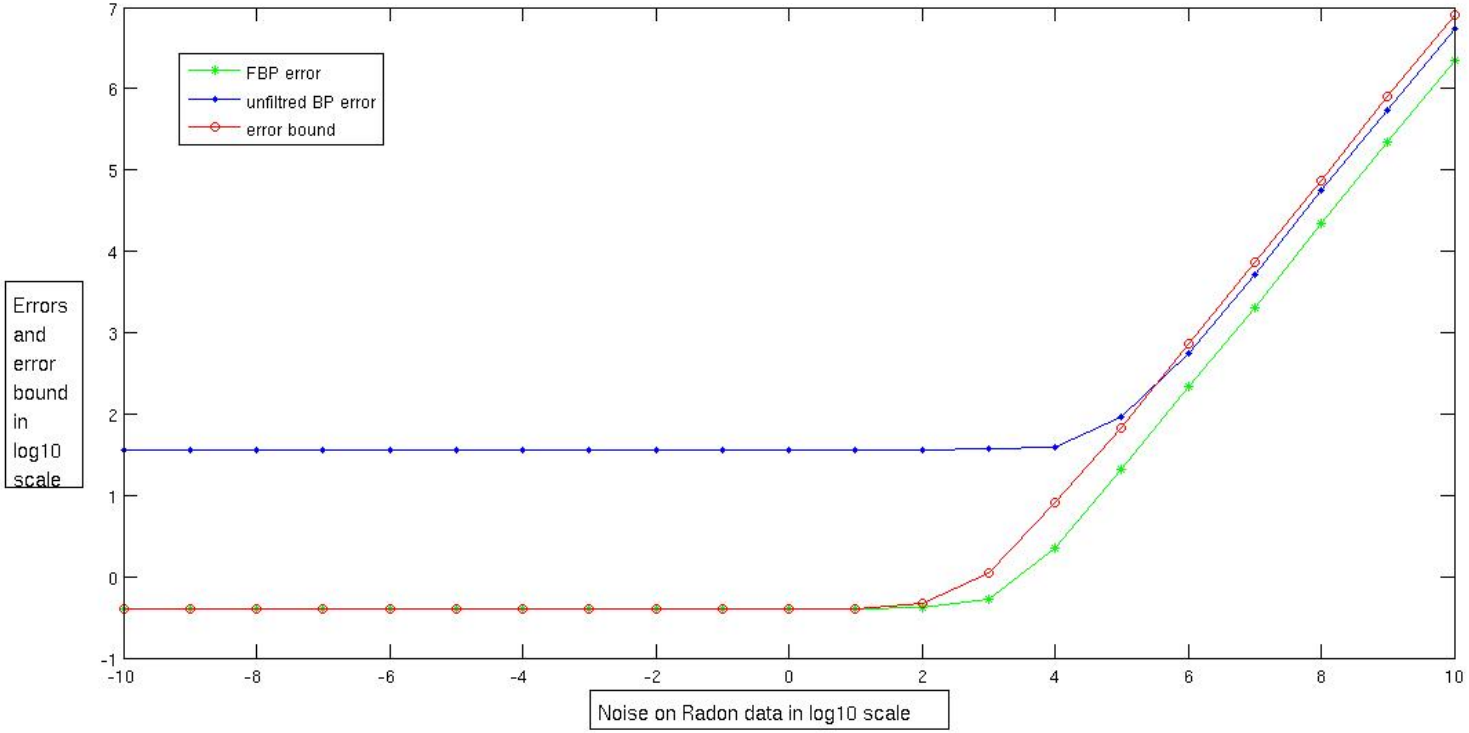


Analytical methods



A test for the error bound using the standard "Ram-Lack" filter.

Essential bibliography

- Feeman, *The mathematics of medical imaging: A beginners guide*, Springer, 2010
- Hansen P.C., Hansen M., *AIR Tools - A MATLAB Package of Algebraic Iterative Reconstruction Methods*, Journal of Computational and Applied Mathematics, 2011
- Natterer F., *The mathematics of computerized tomography*, SIAM: Society for Industrial and Applied Mathematic, 2001
- Sironi A., *Medical Image Reconstruction Using Kernel Based Methods*, Master's Thesis, University of Padova, 2011
- Toft P., *The Radon transform; theory and implementation*, Ph.D. thesis. Department of Mathematical Modeling, Technical University of Denmark, 1996