Reconstruction in X-ray computed tomography

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Calendar of seminars

May 14 Introduction to Analytic Tomography ☑

general introduction, Radon tranform, FBP

May 21 Reconstruction in X-ray computed tomography ☐

iterative methods, kernel methods

May 28 The mathematics behind SPECT/TC reconstruction

analytical and iterative hybrid reconstruction

June 4 Resolution of a Gamma Camera: experimental data and analytical formula

resolution formulas

June 11 An alternative Radon transform for the correction of partial volume effect

a modest proposal..

In the previous seminar we have introduced:

- the Axial Tomography problem.
- the related model: the Radon Transform $\mathcal{R}f = g$.
- a first method for the image reconstruction: FBP.
- the performances of this method.

In this seminar we will talk about the iterative methods for image reconstruction.

Given a basis of functions $\{b_i(\bar{x})\}_{i=1\dots n}$ that interpolates the function f, i.e. such that

$$f(\bar{x}) = \sum_{i=0}^{n} c_i b_i(\bar{x}) \qquad \forall \bar{x} \in X$$

where X is properly chosen, then for the linearity of the Radon transform

$$\mathcal{R}f(\bar{y}) = \sum_{i=0}^{n} c_i \mathcal{R} b_i(\bar{y}) \qquad \forall \bar{y} \in Y = \{(t_j, \theta_j)\}.$$

This is equivalent to the solution of

$$Af = c$$

where $A(i,j) = \mathcal{R}b_i(t_j, \theta_j)$ is a matrix $N^2 \times lp$, f is the unknown vector such that $f_i = f(x_i)$ and c is the vector of the projection data $c_j = \mathcal{R}f(t_j, \theta_j)$.
The methods using this approach are known as Algebraic Reconstruction Techniques or ART.

Using the ART approach we may have to face some problems:

- underdetermined $\longrightarrow$ least squares
- ill-conditioned $\longrightarrow$ regularization
- huge $\longrightarrow$ sparseness

We choose the *natural pixel basis*

$$b_i(x, y) = \chi_{P_i}(x, y).$$

with P_i the i-th pixel of the reconstructed image. We know the Radon transform of each one of these items

$$\mathcal{R}b_i(t, \theta) = meas(\ell_{t,\theta} \cap P_i)$$

where $\ell_{t,\theta} = \{t\bar{\theta} + s\bar{\theta}^{\perp} \mid s \in \mathbb{R}\}$.

Since the matrix A is large and sparse, we can solve the system $Af = c$ by iterative methods.

- the initial vector $f^{(0)}$ is a *blank*.
- the image at step k, $f^{(k)}$, is projected and compared with the data.
- the image is modified considering the error found in the previous step.

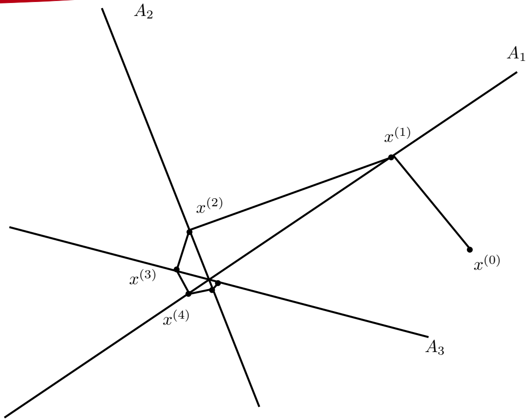
The following methods are the most popular

The Kaczmarz method projects the vector $f^{(k)}$ on k-th row of A.

$$f^{(k+1)} = f^{(k)} + \frac{c_i - A_i^T \cdot f^{(k)}}{A_i^T \cdot A_i} A_i$$

to increase the speed we use

$$f^{(k+1)} = f^{(k)} + \lambda_k \frac{c_i - A_i^T \cdot f^{(k)}}{A_i^T \cdot A_i} A_i$$



$$f^{(k+1)} = f^{(k)} + \frac{c_i - A_i^T \cdot f^{(k)}}{A_i^T \cdot A_i} A_i$$

Maximum Likelihood Expectation Maximization (MLEM) is based on a probabilistic argument (the noise is assumed to be Poissonian).

$$L(f) = P(c|f) = \prod_{i=1}^{lp} \frac{(c_i^*)^{c_i}}{c_i!} e^{-c_i^*}$$

where $c^* = Af$ is the exact sinogram, i.e. the projection of the exact solution f. Equivalently, it maximizes

$$l(f) = \log(L(f)) = \sum_{i=1}^{lp} -(Af)_i + C_i \log(Af)_i + K$$

scheme in a vectorial and multi-step form

1. $c^f := Af^{(k)}$
2. $c^q := c./c^f$ (punctual division)
3. $f^b = A^T c^q$
4. $s_j = \sum_i a_{i,j}$
5. $f^{(k+1)} = f^{(k)}. * f^b./s$ (product and division are made elementwise)

Least Squares Conjugate Gradient (LSCG) is the conjugate gradient method applied to the normal equation of the problem.
Initialization phase:

- given $f^{(0)}$ initial value
- $s^{(0)} = c - Af^{(0)}$
- $r^{(0)} = p^{(0)} = A^T s^{(0)}$
- $q^{(0)} = Ap^{(0)}$

- $\alpha = \frac{|r^{(k)}|^2}{|q^{(k)}|^2}$
- $f^{(k+1)} = f^{(k)} + \alpha p^{(k)}$
- $s^{(k+1)} = s^{(k)} - \alpha q^{(k)}$
- $r^{(k+1)} = A^T s^{(k)}$
- $\beta = \frac{|r^{(k+1)}|^2}{|r^{(k)}|^2}$
- $p^{(k+1)} = r^{(k+1)} + \beta p^{(k)}$
- $q^{(k+1)} = Ap^{(k+1)}$

LSCG example

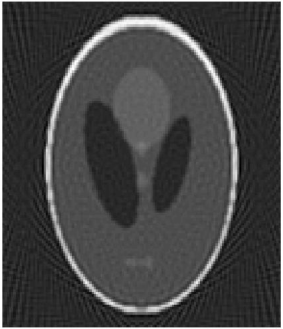
Now instead of the natural pixel basis we can use another basis:

$$b_i(\bar{x}) = B(\bar{x} - \bar{x}_i) = K(\bar{x}, \bar{x}_i)$$

with B a (essentially) compact supported and radial.

Function name	f
Ball	$\chi_{B(\frac{1}{\varepsilon},0)}(r)$
Gaussian	$e^{-\varepsilon^2 r^2}$
Wendland $\varphi_{2,0}$	$(1-\varepsilon r)_+^2$
Wu $\psi_{1,1}$	$(1-\varepsilon r)_+^2(\varepsilon r + 2)$

with $r = \|x\|_2$.



Original phantom and kernel reconstruction with Gaussian kernel and shape parameter $\varepsilon = 1$, after 50 iterations of LSCG.

Lemma

If $\phi(x) = \varphi(\|x\|)$ is a radial function, then its Radon transform $\mathcal{R}f$ is readial, i.e. it depends only on t and it is even.

Theorem

If $\phi(x-y) = K(x,y)$ is a radial function, $\phi \in L^1(\mathbb{R}^d)$, continuous, bounded and positive definite on $\mathbb{R}^2$, then its Radon transform $\mathcal{R}f(t)$ is bounded and positive definite on $\mathbb{R}^1$, provided $\mathcal{R}f \in L^1(\mathbb{R})$.

Theorem (Interpolation error bound for Kernel method)

Let $f \in C_0^\infty(B(0,1))$ a b-band-limited function, and let $g = \mathcal{R}f$ be reliably sampled. Let K be the interpolating Kernel function such that $\mathcal{R}K$ is a symmetric and strictly positive definite kernel and its domain Ω be such that $\partial\Omega$ has regularity at least C^1.
Then there exist positive constants h_0 and $\tilde{C}$ such that, if $h_{X,\Omega} \leq h_0$, then

$$\|f(\cdot) - \sum_{i=0}^{n} c_i K_i(\cdot)\|_{L^\infty} \leq 2|S^1|\; b\sqrt{\frac{N}{18}}\;\tilde{C} h_{X,\Omega} \|\mathcal{R}f\|_{\mathcal{N}_{\mathcal{R}K}(\Omega)}$$

with $h_{X,\Omega}$ the meshsize.

How to choose the shape parameter?

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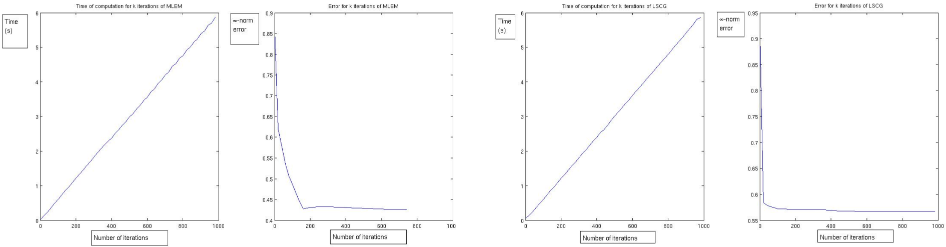
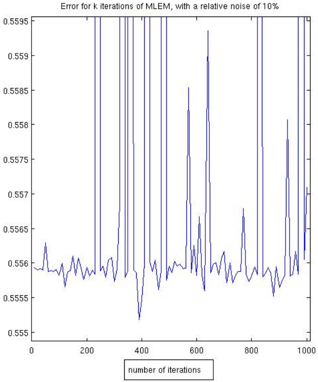
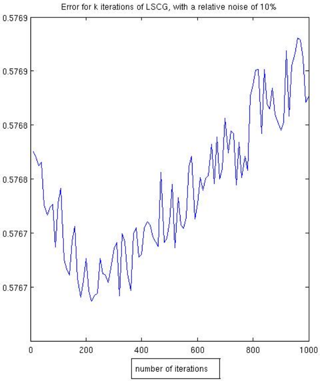
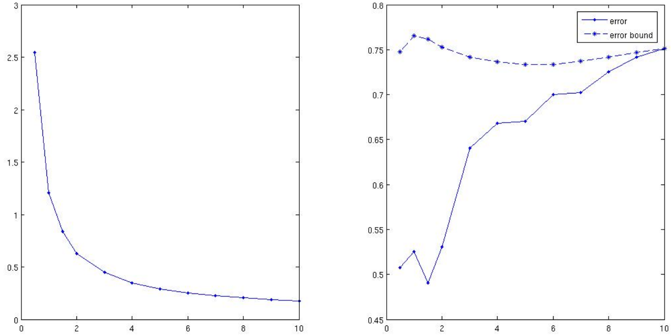


Time and error test for MLEM (upper) and LSCG (lower) at several iterations.



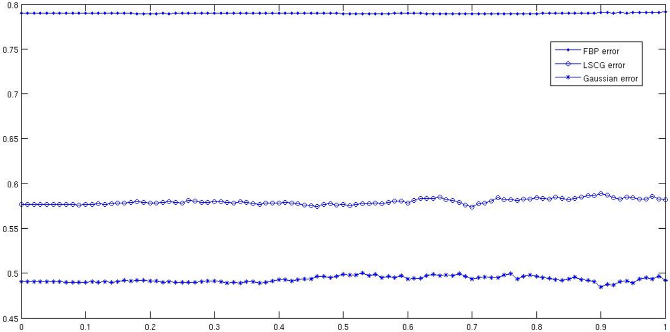


Error of the MLEM and LSCG algorithms with a noise of $\sigma = 10\%$ after several numbers of iterations.

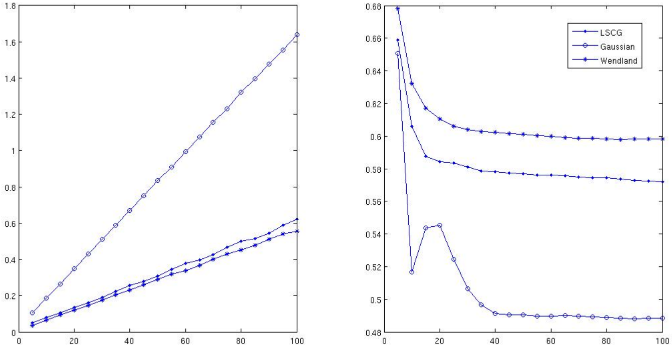


Time of computation in seconds (left), error and error bound (right) in ∞-norm of the kernel method for the Gaussian Kernel with several shape parameters $\varepsilon \in [0.5, 10]$.

Comparison



Error of FBP, LSCG, and Gaussian algorithms as the relative error varies from 0 to 100%.



Computational times in seconds (left) and errors (right) for the LSCG and its kernel versions. Along x we have the number of iterations.

Essential bibliography

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- Hansen P.C., Hansen M., *AIR Tools - A MATLAB Package of Algebraic Iterative Reconstruction Methods*, Journal of Computational and Applied Mathematics, 2011
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- Toft P., *The Radon transform; theory and implementation*, Ph.D. thesis. Department of Mathematical Modeling, Technical University of Denmark, 1996