The mathematics behind SPECT/TC reconstruction

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Calendar of seminars

May 14 Introduction to Analytic Tomography ☑
general introduction, Radon tranform, FBP

May 21 Reconstruction in X-ray computed tomography ☑
iterative methods, kernel methods

May 28 The mathematics behind SPECT/TC reconstruction ☐
analytical and iterative hybrid reconstruction

June 4 Resolution of a Gamma Camera: experimental data and analytical formula
resolution formulas

June 11 An alternative Radon transform for the correction of partial volume effect
a modest proposal..

In the previous seminars we have introduced:

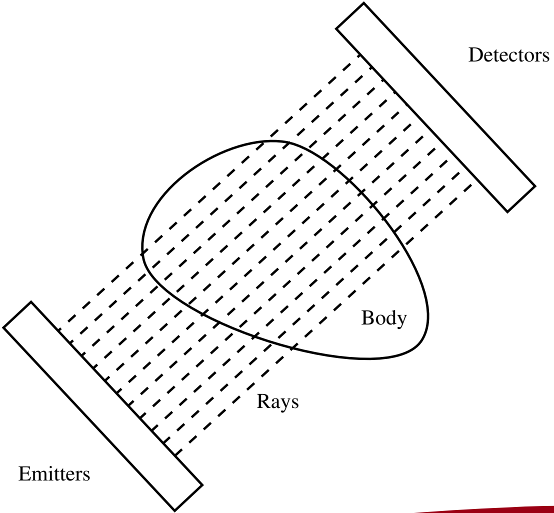
- the Axial Tomography problem.
- the related model: the Radon Transform $\mathcal{R}f = g$.
- some methods for the image reconstruction: FBP, ART.
- the performances of this method.

In this seminar we will talk about the analytical and iterative methods for image reconstruction in the case of hybrid reconstruction (SPECT/CT at first).

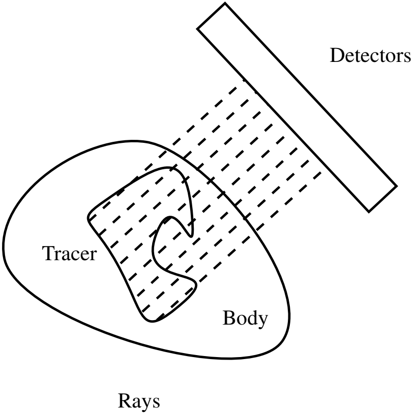
We remind the different kinds of Axial Tomography:

- transmission tomography (e.g. CT): an electromagnetic ray passes trough the patient and is detected at the exit in order to get a morphological analysis of its interior.
- emission tomography (e.g. PET, SPECT): a radioactive tracer is injected into the patient and detected by the machine in order to make an internal functional analysis of the organs.
- hybrid tomography (e.g. SPECT/CT, SPECT/MRI): two simultaneous analysis.

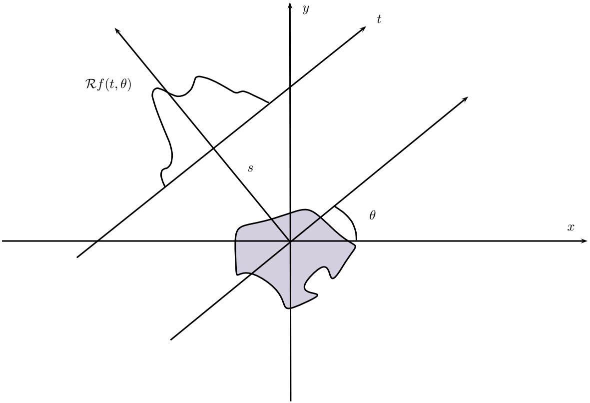
Transmission tomography



Emission tomography



We have already defined the Radon transform



$$\mathcal{R}f(t,\theta) = \int_{\mathbb{R}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds$$

In the case of emission tomography things are more complicated. From Beer's law we obtain that

$$I(x_1) = I(x_0) \exp\left(-\int_{x_0}^{x_1} A(x)dx\right)$$

Suppose to know the attenuation coefficient, say $a(\bar{x})$, we want to obtain the radioactivity $f(\bar{x})$ by its angular projections.

We remind the attenuated Radon transform (briefly AtRT)

$$\mathcal{R}_a f(t,\theta) = \int_{\ell_{(t,\theta)}} e^{-\mathcal{D}a(\bar{x},\theta+\pi)} f(\bar{x})$$

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where $\mathcal{D}$ is the Divergent beam transform defined as follows

$$\mathcal{D}h(\bar{x},\theta) = \int_0^{+\infty} h(x + t\cos\theta, y + t\sin\theta)\; dt = \int_0^{+\infty} h(\bar{x} + t\bar{\theta})\; dt$$

Then in the CT case we have to solve the following problem

RT problem

Given g projection data
find f such that $\mathcal{R}f = g$

while in SPECT case we can approximate f with the solution of the previous problem

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to estimate the attenuation map we may need a simultaneous CT tomography $\longrightarrow$ SPECT/CT.

Also the attenuated transform has an inversion formula. Let by the following definitions:

Definition

Let $g(t)$ be a suitable function, then its Hilbert transform is the function

$$\mathcal{H}g(s) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{g(t)}{s-t} \, dt$$

where the integral a Cauchy principal value.

Definition

Let us define the function

$$h := \frac{1}{2}(I + i\mathcal{H})\mathcal{R}a$$

Theorem (Novikov-Natterer formula)

Let f be a transformable function $g = \mathcal{R}_a f$, and h as in the previous slide. Assume $a(\bar{x})$ known, then f is uniquely determined by the following formula

$$f(\bar{x}) = \frac{1}{4\pi} \mathfrak{Re}\ div \int_{S^1} \theta e^{\mathcal{D}a(\bar{x}, \theta + \frac{\pi}{2})} (e^{-h} \mathcal{H} e^h g)_{(\bar{x} \cdot \bar{\theta}, \theta)}\, d\theta$$

where $S^1 = [0, 2\pi]$.

Attenuation phantom

Activity phantom

Reconstruction of the attenuation map

Reconstruction of the activity

We want to extend the iterative methods to the emission case. Let us consider, as before, the basis $b_i(\bar{x}) = \chi_{P_i}$ and assume

$$f(\bar{x}) = \sum_{i=0}^{n} c_i b_i(\bar{x})$$

then for linearity

$$\mathcal{R}_a f(\bar{y}) = \sum_{i=0}^{n} c_i \mathcal{R}_a b_i(\bar{y}) \qquad \forall \bar{y} \in Y = \{(t_j, \theta_j)\}$$

i.e.

$$Bc = d$$

where $B_{i,j} = \mathcal{R}_a b_i(t_j, \theta_j)$, $c_i = f(\bar{x}_i)$ the unknown term and $d_j = \mathcal{R}_a f(t_j, \theta_j)$ the data.

In order to compute the matrix B we have to consider the attenuation coefficients in the natural pixel basis

$$a(\bar{x}) = \sum_{k=1}^{N^2} g_k \chi_{P_k}(\bar{x}).$$

According to Beer's law

$$I_{out} = I_{in} \exp\left(-\sum_{k=1}^{N^2} g_k \ \ meas(P_k \cap \ell^{+}_{\bar{x},\theta}) \right).$$

Now, if we consider the matrix A used in CT tomography we can compute the outgoing rays from the pixel P_i in (t_j, θ_j) as

$$B_{i,j} = A_{i,j} \ \exp\left(- \sum_{(k_1,k_2)\in K_{(i,j)}} g_k \ \ meas(P_k \cap \ell_{t_j,\theta_j}) \right) =$$

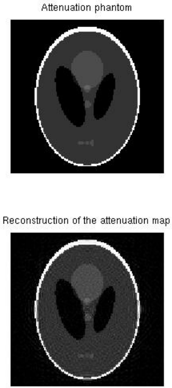
$$A_{i,j} \ \exp\left(- \sum_{(k_1,k_2)\in K_{(i,j)}} g_k \ A_{k_1,k_2} \right)$$

where $K_{(i,j)} = \{(k_1, k_2)\} \subset \{1, \dots, N^2\}^2$ is the set s.t. $k = lp - ((k_1 - 1)p + k_2) + 1$ are the indexes of the pixels covered by the line $\ell^{+}_{\bar{x}_i,\theta_j}$.

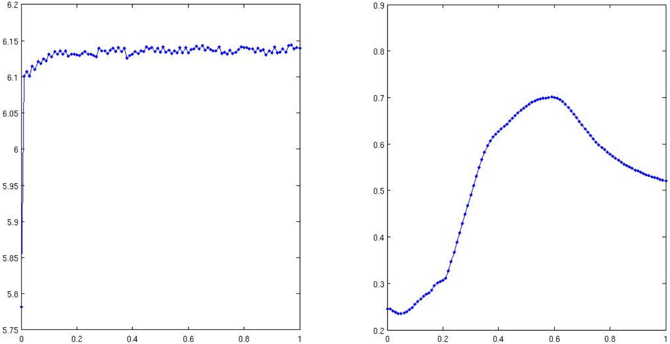
We can introduce a relaxation parameter $\lambda \in [0, 1]$ to weight the effect of the attenuation

$$B_{i,j}^{(\lambda)} = A_{i,j} \ \exp \left(-\lambda \sum_{(k_1,k_2) \in K_{(i,j)}} g_k \ A_{k_1,k_2} \right)$$

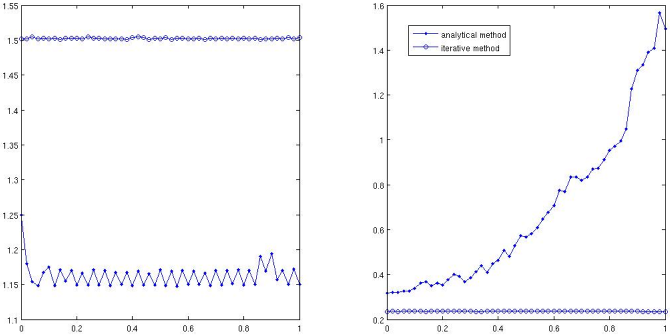
Note that $B^{(0)} = A$ and $B^{(1)} = B$.
We observe that with this little change the linear system is more accurate.



Analytical reconstruction of a SPECT/CT phantom data with $\lambda = 0.1$.



Trial-and-error test for the λ parameter. To the left we see the total computational time of the matrix B (in seconds) and the solution of the system. Right: the errors for various values of $\lambda \in [0, 1]$.



Time of resolution in seconds (left) and error (right) for the analytical and iterative methods for the resolution of the hybrid SPECT/CT simulated problem at several relative noise levels form 0 to 100%.

Essential bibliography

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