Physical Basis of Magnetic Resonance Imaging

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Calendar of seminars

Sept. 19 **Physical Basis of Magnetic Resonance Imaging** ☐

spin under a magnetic field, RF pulses, Bloch equations

Sept.26 **Sequences and Reconstruction in MRI** ☐

K-space, aliasing, sequences

Oct. 10 **Positron Emission Tomography: an introduction** ☐

general introduction, attenuation correction, level set methods

Oct. 17 **Kinetics of the Tracer in PET** ☐

compartment model for ^{18}FdG

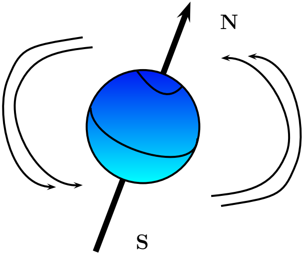
Oct. 31 **Dual-Modality Imaging** ☐

Magnetic Resonance Imaging (MRI), elderly known as Nuclear Magnetic Resonance (nMR) is an *in vivo* medical imaging system that uses the resonance of magnetic fields. This exam is non-invasive and much secure compared to the analog X-ray tomography.

In fact the electromagnetic rays in use have a maximum frequency of about 10^5Hz (which is in the order of radio waves), which is much less than the X-ray frequency, about 10^{16}Hz.

The nucleus of an atom has a property called spin. This means that we can consider an atom as a spinning wheel.

This property leads also to an electromagnetic effect: each nucleus generates a small magnetic field. This means that we can also think of nuclei as a magnet.



For our purposes we will consider the nuclei of hydrogen (H), because is has an odd number of protons and so the magnetic moment is not null and also because is very common.

A proton p has an intrinsic angular momentum J_p and a magnetic momentum μ_p. There exists a direct proportion between these two quantities

$$\mu_p = \gamma_p J_p$$

where γ_p is a constant called gyro-magnetic ratio and it is related to the kind of atom in examination. For instance the hydrogen has a ratio equal to $\gamma_H \cong 2.68\ T^{-1}s^{-1}$. To be more precise the gyro-magnetic ratio depends on the chemical bound of the atom

$$\gamma_p = (1 - \sigma)\gamma$$

where the constant $\sigma \in [10^{-6}, 10^{-4}]$ is called chemical shift.

Now if we turn on a stationary and uniform magnetic field $\mathbf{B_0}$, the quantum mechanic expected value of the intrinsic angular momentum will satisfy the law

$$\frac{d < J_p >}{dt} = < \mu_p > \times \mathbf{B_0}$$

so, given the proportion we obtain

$$\frac{d < \mu_p >}{dt} = \gamma < \mu_p > \times \mathbf{B_0}$$

Since we can not measure the momentum of every single atom, we define the (total) magnetic momentum **M** as

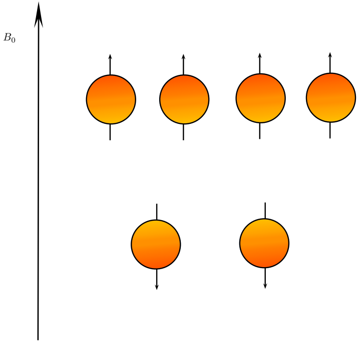
$$\mathbf{M} = \sum_{p} \mu_{p}$$

then we can formulate the differential equation

$$\dot{\mathbf{M}} = \gamma \mathbf{M} \times \mathbf{B_0} \tag{1}$$

which will be fundamental for our purposes.
Even if the nuclei are under a quantum effect, a classical approach will be more feasible and will not lead to great errors.

In absence of an external magnetic field the nuclei arrange themselves randomly and their momentums mutually annul, so $\mathbf{M} = 0$. Since the magnetic field of the Earth is weak (from 2 to 7 $\cdot 10^{-5}$ T), we can consider that a normal body has a magnetic momentum $\mathbf{M} \approx 0$.
Otherwise, if we turn on a stationary magnetic field along the z-axis $\mathbf{B_0} = (0, 0, B_0)$, the nuclei will show a sort of herd mentality. In fact their rotation axes arrange themselves parallel to the axis z. Some of them align themselves with the same verse of $\mathbf{B_0}$ (low energy), and some others will take the opposite direction (high energy). As we will see in the next section, the most part of the nuclei will arrange in a low energy position.



A more accurate description of the motion reveals that the axes has a precession along the direction of the external field at the same frequency ω_0, called the Larmor frequency, but with different phases.

The precession tends to the equilibrium $(0, 0, M_{eq})$ exponentially, with a recovery time T_1 (spin-lattice) along the axis z, and with a loss time T_2 (spin-spin) along the plane xy. These two times are different because of the different phase of the nuclei.
The parameters T_1 and T_2 depends from the chemical bound of the Hydrogen molecules, then from the tissue.
The main problem we will face is that the precession is not visible from the total momentum because the convergence to the equilibrium is too fast.

The disposition of the nuclei in the verse of the magnetic field causes the precession motion to be difficultly measurable. To make the precession visible we have to study deeper, the alignment to the axis of the nuclei. According to the Boltzmann factor the probability that a nucleus arranges himself along the z-axis is equal to the negative exponential of the ratio between magnetic and thermal energy,

$$P(E_i) = \exp\left(-\frac{E_{magn}}{E_{therm}}\right) = \exp\left(-\frac{\gamma B_0 \hbar}{k_b T_K}\right)$$

where $\hbar \cong 1.05 \cdot 10^{-34}\ Js$ is the normalized Plank constant, $k_b \cong 1.38 \cdot 10^{-23}\ JK^{-1}$ is the Boltzmann constant and T_K is the absolute temperature in Kelvin degrees.

Let us calculate this value for a typical MRI experiment, $B_0 = 1T$, $T = 300K$.

$$P(E_i) \cong \exp\left(-\frac{3\, s^{-1} T^{-1} \cdot 1\ T \cdot 10^{-34}\ Js}{1 \cdot 10^{-23}\ JK^{-1} \cdot 3 \cdot 10^2 K}\right) = \exp\left(-10^{-13}\right)$$

for the first order Taylor approximation we can compute

$$P(E_i) \cong 1 - 10^{-13}.$$

This means that only one atom over 10^{13} chooses a high energy position, and the magnetic momentum tends quickly to the z-axis. From the Boltzmann factor comes an idea to make more visible the precession: we can decrease the room temperature or increase the external magnetic field.
For having a visible change we have to move two parameters in values that are physically unacceptable. In conclusion we have to find another way to make the precession measurable.

To move down the magnetic momentum we have to resort to a Radio-frequency pulse (RF-pulse), a second magnetic field $\mathbf{B_1}$ of a minor intensity than the static field and rotating along the plane *xy* at *exactly* the Larmor frequency ω_0. In this way a physical phenomenon will occur, the magnetic resonance. In fact a huge number of nuclei will *gain energy* and and the magnetic momentum $\mathbf{M}$ will separate from the z-axis.
Moreover with the RF field turned on the phase of the precession will align, while when we turn off the RF field there will be “dephasing”.

To see mathematically phenomenons described in the previous section, we need a toy-model, a set of mathematical equations that describes the motion of the aggregate magnetic momentum $\mathbf{M} = (M_x, M_y, M_z)$. So let us consider the equation (1) applied to an external field $\mathbf{B}$ and study the solution of these equation for several external fields.

Let us consider the case that only the static field along the axis z is on. In this case we set $\mathbf{B} = (0, 0, B_0)$. If we add a relaxation term that describes the polarization along the axis z (T_1), and the decay along the xy-plane (T_2) we get the **Bloch equation**

$$\dot{\mathbf{M}} = \gamma \mathbf{M} \times \mathbf{B} - \left(\frac{M_x}{T_2}, \frac{M_y}{T2}, \frac{M_z - M_{eq}}{T_1} \right) \qquad \text{(Bloch-1)}$$

we can write this equation component-wise

$$\begin{cases} \dot{M}_x &= M_y B_0 - \frac{M_x}{T_2} \\ \dot{M}_y &= M_x B_0 - \frac{M_y}{T_2} \\ \dot{M}_z &= -\frac{M_z - M_{eq}}{T_1} \end{cases} \qquad (2)$$

The solution to this ODE is

$$\left\{\begin{array}{lcl} M_x & = & e^{-t/T_2}(M_{x,0}\cos\omega_0 t - M_{y,0}\sin\omega_0 t) \\ M_y & = & e^{-t/T_2}(M_{x,0}\sin\omega_0 t + M_{y,0}\cos\omega_0 t) \\ M_z & = & M_{z,0}\, e^{-t/T_1} + M_{eq}(1 - e^{-t/T_1}) \end{array}\right. \tag{3}$$

where $\omega_0 = -\gamma B_0$ is the Larmor frequency. This is clearly a precession along the $z-$axis at the Larmor frequency, and the solution tends to the equilibrium $(0, 0, M_{eq})$ in a exponential way, so the precession is very hard to see experimentally.

We turn on a RF pulse, rotating with angular speed ω in the plane xy, we now have $\mathbf{B} = (B_1 \cos \omega t, B_1 \sin \omega t, B_0)$. Since we consider what happens while the pulse is on, we can ignore the relaxation terms and write

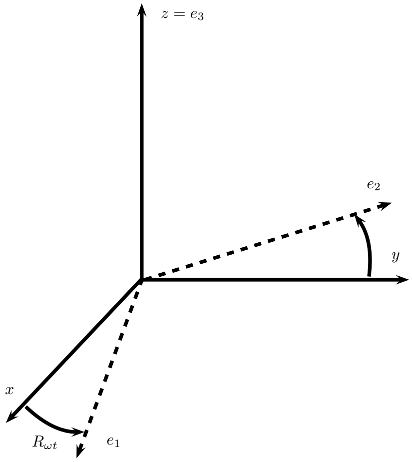
$$\dot{\mathbf{M}} = \gamma \mathbf{M} \times \mathbf{B} \tag{4}$$

again, we rewrite the equation element-by-element

$$\begin{cases} \dot{M}_x &= \gamma \left(M_y B_0 - M_z B_1 \sin \omega t \right) \\ \dot{M}_y &= \gamma \left(M_z B_1 \cos \omega t - M_x B_0 \right) \\ \dot{M}_z &= \gamma \left(M_x B_1 \sin \omega t - M_z B_1 \cos \omega t \right) \end{cases} \tag{5}$$

now, for simplicity, we study the system in a reference frame rotating along the axis z like the RF pulse.

$$\left\{ \begin{array}{lclll} \mathbf{e1} & = & (\cos \omega t, & \sin \omega t, & 0) \\ \mathbf{e2} & = & (-\sin \omega t, & \cos \omega t, & 0) \\ \mathbf{e3} & = & (0, & 0, & 1) \end{array} \right. \tag{6}$$



The rotating non-inertial frame $\{e_1, e_2, e_3\}$

Now if we consider the functions

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = R_{\omega t} \begin{pmatrix} M_x(t) \\ M_y(t) \end{pmatrix} \tag{7}$$

where $R_{\omega t}$ is the matrix of rotation of angle ωt.
Substituting the functions (7) in the system (5) we get

$$\begin{cases} \dot{u} & = & (\gamma B_0 + \omega)\, v \\ \dot{v} & = & -\,(\gamma B_0 + \omega)\, u + \gamma B_1 M_z \\ \dot{M}_z & = & -\gamma B_1 v \end{cases} \tag{8}$$

now we observe that if the angular speed of the RF pulse is exactly the Larmor frequency $\omega = \omega_0 = -\gamma B_0$ the system is simplified and we obtain the second version of Bloch equation

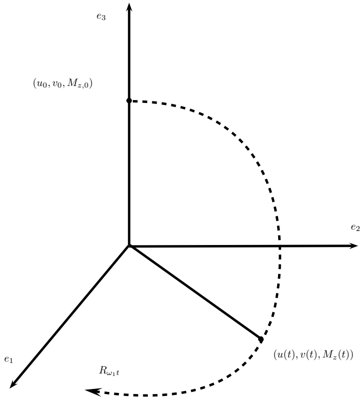
$$\begin{cases} \dot{u} & = & 0 \\ \dot{v} & = & \gamma B_1 M_z \\ \dot{M}_z & = & -\gamma B_1 v \end{cases} \qquad \text{(Bloch-2)}$$

this means that the system is affected by resonance, since and the aggregate momentum tends to go move down from the equilibrium position. This result is known as Larmor Theorem.

The solution of this equation is simple

$$\left\{\begin{array}{lcl} u & = & u_0 \\ v & = & v_0 \cos\omega_1 t - M_{z,0}\sin\omega_1 t \\ M_z & = & v_0 \sin\omega_1 t + M_{z,0}\cos\omega_1 t \end{array}\right. \tag{9}$$

where, as before $\omega_1 = -\gamma B_1$.



The solution of the second Bloch equation in the non-inertial reference frame.

The motion in inertial reference frame is the composition of two circular uniform motions in orthogonal directions, which means that we have a uniform helical motion over a sphere.

For reasons that we will soon clarify, sometimes in addiction to the RF pulse we also need a gradient pulse, another magnetic field along z which has the form $\mathbf{B_G} = (0, 0, B_g) = (0, 0, G \cdot x)$, where, for instance, G is a constant vector of $\mathbb{R}^3$. Now we rewrite the eq. (Bloch-2) considering a $B_0 + B_g$ instead of B_0 and we obtain

$$\begin{cases} \dot{u} & = & \gamma B_g v \\ \dot{v} & = & -\gamma B_g v + \gamma B_1 M_z \\ \dot{M}_z & = & -\gamma B_1 v \end{cases} \tag{10}$$

Since $B_1 \ll B_0, B_g$ we can ignore this term and thus obtain the third Bloch equation:

$$\begin{cases} \dot{u} &= \gamma B_g v \\ \dot{v} &= -\gamma B_g v \\ \dot{M}_z &= 0 \end{cases} \tag{Bloch-3}$$

the solution to this ODE is similar to the previous

$$\begin{cases} u &= u_0 \cos \omega_g t - v_0 \sin \omega_g t \\ v &= u_0 \sin \omega_g t + v_0 \cos \omega_g t \\ M_z &= M_{z,0} \end{cases} \tag{11}$$

where, as one can guess, $\omega_g = -\gamma B_g$.

Essential bibliography

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