Positron Emission Tomography: an introduction

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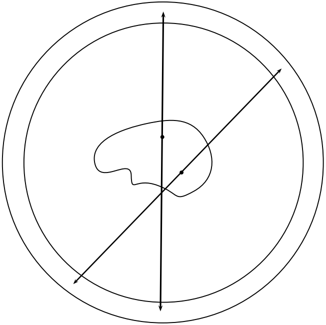
Calendar of seminars

- **Sept. 19** Physical Basis of Magnetic Resonance Imaging ☑
  spin under a magnetic field, RF pulses, Bloch equations
- **Sept.27** Sequences and Reconstruction in MRI ☑
  K-space, aliasing, sequences
- **Oct. 18** Positron Emission Tomography: an introduction ☐
  general introduction, attenuation correction, level set methods
- **Oct. 31** Kinetics of the Tracer in PET ☐
  compartment model for ^{18}FdG
- **Nov. 14** Dual-Modality Imaging ☐

Positron emission tomography (PET) is a nuclear medical imaging technique that produces an image of functional processes in the body. The PET machine detects pairs of gamma rays, or positrons emitted by the annihilation of a tracer (as the ^{18}FdG), introduced into the body.



The particularity of PET stands in the possibility of recover a sequence of images as a time series. This allows the researchers to study the evolution of the tracer kinetics, which will be treated in the next seminar.
Since the positrons are emitted two by two, before considering a couple of data as the product of the same annihilation process, the machine filters only the pairs of positrons incoming in the same (small) temporal window $\triangle t$. This allows to reduce the effect of scattering, so the data collected after this filtering is quiet reliable.

Let us introduce the mathematical models for the PET scan.
We recall the Radon transform used in CT

$$\mathcal{R}f(t,\theta) = \int_{\ell_{t,\theta}} f(\bar{x})\, d\bar{x}$$

and the attenuated Radon transform used in SPECT

$$\mathcal{R}_a f(t,\theta) = \int_{\ell_{t,\theta}} \exp\left(-\int_{\ell^+_{t,\theta}(\bar{x})} a(\bar{y})\, d\bar{y}\right) f(\bar{x})\, d\bar{x}$$

in the PET case we receive two signals in the opposite direction ℓ^+ and ℓ^-, belonging to the same line ℓ. so in that case the integral of a does not depend on $\bar{x}$, and then we can write

$$\widetilde{\mathcal{R}}_a f(t,\theta) = \exp\left(-\mathcal{R}a(t,\theta)\right)\mathcal{R}f(t,\theta)$$

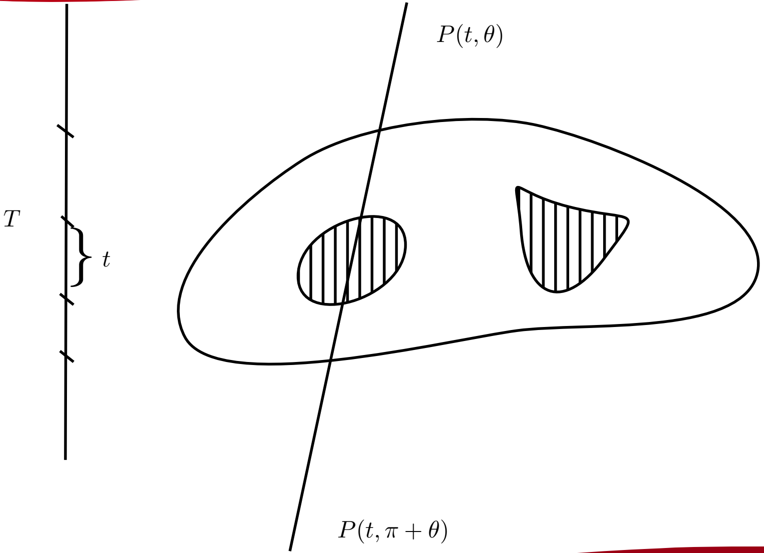
this fact allows to correct the attenuation with ease.
Since the image reconstruction techniques are similar to the ones used in CT and SPECT, in this seminar we shall only treat some post-processing methods as the attenuation correction and the segmentation.

Soreson proposed a pre-correction of the data obtained from the projections. It is hypothesized the uniform distribution of the tracer and the uniform density of the tissue along each ray; this means that the algorithm will give out good results only when the area to reconstruct is sufficiently uniform, for example the brain. Let $P_{k,\theta}$ and $P_{k,\theta+\pi}$ two opposite projections, T the diameter body and t the diameter of the active region along the straight line projection.

Then, if we call $N = \sqrt{P_{k,\theta} \cdot P_{k,\theta+\pi}}$, the geometric mean of the data the correct value for both angles is

$$N = N_0 e^{\gamma \frac{T}{2}} \left(\frac{f \gamma \frac{T}{2}}{\sinh(f \gamma \frac{T}{2})} \right)$$

where γ is the linear attenuation coefficient (e.g. the brain, if the tracer used and ^{99m}T we suppose $\gamma = 0,035 cm^{-1}$) and $f = \frac{t}{T}$ is a factor which in practice varies from 0.5 to 0.75. We can fix that value, for instance to 0.6 or better to an average estimated value of $\frac{t}{T}$.


$P(t,\theta)$
T
t
$P(t,\pi+\theta)$

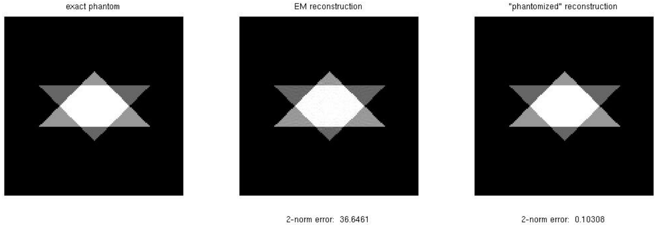
The Chang algorithm makes a post-correction on the reconstructed image. The result of the classic reconstruction is multiplied pixel by pixel by the weights obtained from an estimate of the mean attenuation that the beam with angle θ had been subject to at the transition from pixel (x, y).
Then, if P is the image obtained e.g. by FBP and P_0 the corrected image, then $P_0(x, y) = c(x, y)P(x, y)$

The correction terms are:

$$C(x,y) = \left(\frac{1}{I} \sum_{i=1}^{I} e^{-\mu l_{\theta_i}} \right)^{-1}$$

where l_{θ_i} is the distance between the pixel (x, y) and the edge of the body at the angle θ_i and μ is the linear attenuation coefficient, supposed to be constant.
Therefore also the Chang algorithm gives good results if the tested area is uniform.

The aim of segmentation algorithms is to reduce an image on the form $f = \sum_i \chi_{B_i}$ with $B_i \subset \mathbb{R}^2$ some compact sets. This type of functions is congenial to the reconstruction because it splits the region of interest (or ROI) in different areas where the attenuation coefficient is constant. We can assign these constants if we recognize the type of tissue or we can calculate them. We need a segmentation algorithm that levels the image obtained by the image reconstruction.



The proposed "naïve" algorithm is the following:

1. P is the $N \times N$ matrix of reconstructed image, k the number of different levels that you want to look in $[0, 1]$
2. put its elements in a column vector, $P = P(:)$ long N^2
3. for $i = 1, \ldots, N^2 - 2$
    1. we calculate S the sum and D the difference of the elements in $X = P(i : i + 2)$
    2. if S or D has norm less than $\frac{2}{k}$ then $P(i : i + 2) = mean(X)$ were *mean* is the the average
    3. otherwise P remains the same
    4. end of the *for* loop
4. repeat 2. and 3. for each rotation of $\frac{\pi}{2}$ of the matrix P

We now introduce Level Set methods, which will be useful both for the shape recognition and (by consequence) for the segmentation. As seen in [4], the main idea is to represent a closed curve of the space Γ as the zero-level set of a function $\Gamma = \left\{ \mathbf{x} \in \mathbb{R}^2 \text{ s.t. } \phi(\mathbf{x}) = 0 \right\}$.
We add a "time" dimension to the function in order to control the evolution of the curve

$$\Gamma(t) = \left\{ \mathbf{x} \in \mathbb{R}^2 \text{ s.t. } \phi(\mathbf{x}, t) = 0 \right\}$$

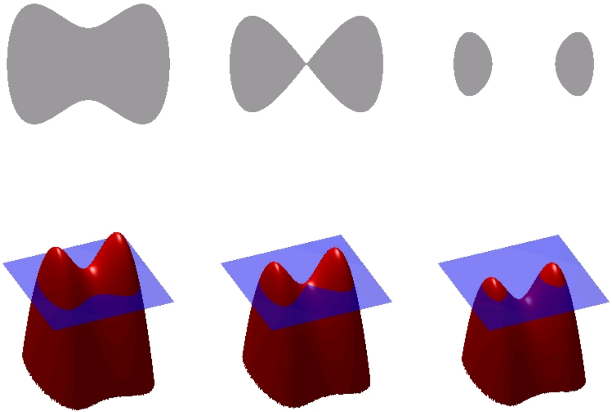


Image courtesy of Wikipedia.

Now for the chain rule we get

$$\frac{\partial}{\partial t}\phi + \nabla_{\mathbf{x}}\phi(\mathbf{x}(t), t) \cdot \dot{\mathbf{x}} = 0$$

since we want the curve to expand to the normal direction $\mathbf{n} = \frac{\nabla_{\mathbf{x}}\phi}{|\nabla_{\mathbf{x}}\phi|}$ with constant speed we get the differential equations

$$\frac{\partial}{\partial t}\phi + F|\nabla_{\mathbf{x}}\phi| = 0$$

with $F = \mathbf{n} \cdot \dot{\mathbf{x}}$ a positive constant.

An adaptation of the Level Set method for post-processing PET images, proposed in [1], is based on the MLEM reconstruction algorithm. Let the reconstruction equation be $Af = c$ with f the vector of the reconstructed image values, c the raw projection data and A the ART matrix, as in the past seminars.
MLEM algorithms finds the solution that minimize the functional

$$L(f) = \sum f_i - \sum c_i \log(Af)_i + V(f)$$

where $V(f)$ is a suitable regularization term.

Case of a unique curve

Consider Γ closed curve as the zero-level-set of the signed distance function

$$\phi(x) = \begin{cases} d(\Gamma, x) & x \in int(\Gamma) \\ -d(\Gamma, x) & x \in ext(\Gamma) \end{cases}$$

we suppose that the image takes constant values inside and outside Γ, i.e.

$$f(x) = k_1 H(\phi(x)) + k_2 (1 - H(\phi(x)))$$

where H is the Heaviside function so that $H(\phi(x)) = \chi_{int(\Gamma)}(x)$.

Case of n closed curves

If we have a number of closed curve it is a bit more difficult. For $i = 1, \ldots, n$, let the binary representation of $i - 1$ be $bin(i-1) = (b_1^i, \ldots, b_n^j)$. Then f can be written as

$$f(x) = \sum_{i=1}^{2^n} k_i \prod_{j=1}^{n} R_i(\phi_j(x))$$

where

$$R_i(\phi_j) = \begin{cases} H(\phi_j) & \text{if } b_j^i = 0 \\ 1 - H(\phi_j) & \text{if } b_j^i = 1 \end{cases}$$

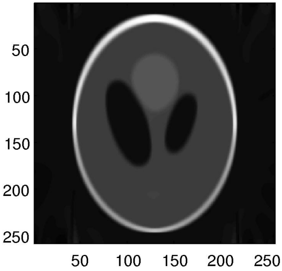
We calculate with the chain rule $\frac{\partial L}{\partial \phi_j} = \frac{\partial L}{f} \frac{\partial f}{\partial \phi_j}$.
We easily see that $\frac{\partial L}{\partial f} = \mathbf{e} - A^T(b./Af)$ plus the derivative of the regularization term. As regularization term we use the length of the curves $V(\phi_j) = \int |\nabla H(\phi_j)|$ whose derivative is known

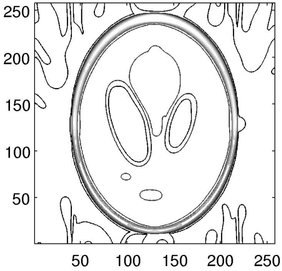
$$\frac{\partial V}{\partial \phi_j} = -\nabla \cdot \left(\frac{\nabla \phi_j}{|\nabla \phi_j|} \right) \delta(\phi_j).$$

With these tools we can compute $\frac{\partial L}{\partial \phi_j}$, and use it in the following iterative algorithm:

1. Choose initial level sets $\phi_i^{(0)}$ and 'time step' $\triangle t$.
2. Update the level sets $\phi_i^{(n+1)} = \phi_i^{(n)} - \triangle t \frac{\partial L}{\partial \phi_j}$
3. Reinitialize the level sets.

Some examples in Matlab.





Essential bibliography

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